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THEORETICAL INVESTIGATION OF DUCTED PROPELLER AERODYNAMICS

by

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Volume I

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INTRODUCTION

and

GENERAL COMMENTS

This introduction will review the general status of the problem of the design of the vertical take-off airplane. It will attempt to indicate the most effective means, both theoretical and experimental, to further the knowledge of the problem. There exist certain basic requirements which cannot be bypassed and without the full understanding of which a really successful design cannot be produced. The essential elements necessary for the development of such a design will be mentioned briefly.

The first matter to be noted is that the most costly method of creating lift is that of the relatively heavily loaded fan. For this reason it is basically desirable to equip the aircraft also with a regular wing surface of sufficient area to relieve the fan-produced lift at the lowest forward velocity consistent with the particular mission. As the forward speed is increased a larger and larger fraction of the required fan power may be employed for forward propulsion, either by direct mechanical transmission to a conventional propeller or by merely directing the jets gradually more in a rearward direction. In the latter case there are several possibilities. One may, for instance, change the direction of the jet and the cross section of the outlet openings in such a manner that the area of the opening is reduced as the cosinus of the deflection angle from the vertical. The mass of air per second is then reduced while the pressure head of the fan is increased so as to keep the horsepower requirement constant, using the excess power for acceleration. This calls for an adjustable pitch propeller which, of course, cannot cover too much of a range. This case will, however, be covered in the following as a simple example.

A more desirable case is, however, one in which the airflow is kept constant in forward flight by employing a programmed outlet cross section as the direction of the thrust vector is gradually inclined backwards. Very simply, the pressure head behind the propeller is then maintained at a level to balance the velocity head of the incoming air, due to the forward velocity. The propeller or fan is, in this case, always working at optimum or design condition and is of fixed angle of attack, while only the outlet is adjustable. This design is novel as far as is known.

The primary problem concerns the use of ducted propellers. It has and can be shown that a ducted propeller carries, in the extreme ideal case of proper inlet design, as little as one half of the total thrust. The question is whether this gain is of any realistic or practical value.

The work, of course, is done exclusively by the propeller which then carries one half thrust at double velocity through the disk. However, unless the converging inlet is much larger than the propeller area and scientifically designed, the actual effect is found to be only in the order of 10 - 20% . Also, this effect is not one of efficiency but rather one relating to propeller size and load. The advance ratio of the propeller is doubled, which is favorable. It is definitely to be expected that the overall efficiency in lifting effect generally is lower for a ducted propeller than for a free propeller. Neither may there be any net weight saving, in fact, the gain may only be one of a reduction in the required gear ratio; while the added surface contributes to increase in weight and drag.

In forward flight the angular deflection of the airflow entering into the inlet opening of the duct is on the average near 90° , or a right angle. It is quite evident that a 90° flow deflection, if unaided, will result in a completely turbulent flow unless the forward velocity is very small compared to the inlet velocity. Also, the velocity at the leading edge side of the inlet will be very much larger than the inlet velocity near the rear side. The center of thrust of the propeller will move both rearward and towards the advancing side of the propeller as in the case of an "unducted" propeller in a free stream (as demonstrated in the NASA wind tunnel test referred to elsewhere in this report). This unbalance of the propeller thrust may have disasterous effect on the propeller; the gears and bearings will require a heavier construction to take the vibration load on blades and bearings.

Therefore, to obtain proper design condition, care must be taken that the flow enters the duct at constant velocity around the circumference of the duct inlet in order to center the load and to obtain full efficiency of the propeller. The mathematical problem of the proper duct design to fulfill these conditions is given in the body of this report. This study should be followed up with basic experiments on the inflow problem and the effect of baffles as a first step.

In regard to the outlet design, it has already been pointed out that it is highly desirable to maintain a constant air velocity past the propeller or fan. The cross section of the outlet duct measured perpendicularly to the velocity vector must then be decreased slightly with forward speed to maintain a constant pressure differential across the fan at all speeds. The theory and design of such a duct should be fully developed and elementary tests should again be performed as the primary step.

The discussion so far relates only to a design in which the fan is installed in the wing or body of the aircraft with the fan axis generally perpendicular to the flight direction or with a slightly rear-ward directed thrust axis. In this case, a relatively large fan area may be most easily accommodated and, in fact, as little as one half of the thrust need, in the limit, be carried by the fan in hovering condition. However, scientifically designed inlet and outlet vanes must be provided to avoid deterioration of the lifting effect in forward transition and to adjust the thrust vector as required in magnitude and direction.

Another method is the use of fans turnable around a transverse diameter of the fan disk (Doak type). In this case, the duct itself is of little use since it must be reasonably streamlined and thus carries only a small fraction of the total thrust. It can be shown that a slight increase in solidity of the propeller is more efficient unless the propeller is already very heavily loaded. Strictly speaking, inlet vanes are again required in transition since the propeller efficiency will suffer by the fact that the thrust vector moves backwards and towards the advancing side of the propeller disk. In favor of this design is the fact that the fan is only exposed to heavy vibrational forces in the transition, and may be adjusted properly in full forward flight with all, or almost all, lift transferred to the wing.

Another interesting possibility is the arrangement tested in NACA TN 3198 described under the title - "Dynamic Stability and Control Characteristics of a Cascade-Wing Vertically Rising Airplane Models in Take-Offs, Landings, and Hovering Flight" by M. O. McKinney, Louis P. Tosti, and Edwin E. Davenport (June, 1954). In this experiment the propeller or propellers are located with the axis in the normal fore-and-aft position, the slipstream being deflected by outlet vanes only. Although these vanes were not designed for a (slightly) convergent flow and there was, in consequence, some stall flutter causing a sustained pitching oscillation, the net result was promising and should be fully explored in the near future. The overall lifting efficiency of the fans reached 92% which efficiency with correction for scale effect and design using a slightly convergent flow in the turning vanes would have reached at least 4% more in a full-scale design. How the practical design of such aircraft can be accomplished with adjustable or removable vanes is a problem that might be fully justified as worthy of further work. The efficiency of this arrangement is apparently higher than in any other proposed solution with or without ducted fans. The practical problems in such a design appear perfectly solvable on the basis of adequate experimentation.

We shall finally present a few remarks about the theoretical and actual effect of a duct since there is, sometimes, a certain amount of misunderstanding. It has been stated and shown in the theoretical section of this report, that the duct, when inserted in a large surface such as a wing and with a scientific design of optimum inlet configuration, may carry in the limit one half of the total thrust of the combined arrangement. The propeller may, in consequence, be of smaller diameter and run at a more favorable advance ratio. However, there exists the fundamental difficulty that there tends to be a cancelling effect of the extraneous lift caused by a considerable suction effect on the lower surface which is inherent in the problem. This can theoretically be avoided by a cylindrical extension of the duct below the wing surface. The lifting efficiency, therefore, always appears to fall short of expected values even in hovering flight. In general, a duct design can evidently only be defended in the case in which the wing is used to enclose the fan for flight at very high speeds and for a very heavily loaded fan. For lower top speeds a design a la NACA or Doak may be more promising, subject to necessary tentative or obvious improvements already indicated.

We shall finally give a few remarks about the pitching moment. There is again a certain misunderstanding existing in regard to this problem. This pitching moment is <u>not</u> an inherent fault of the design and it <u>cannot be alleviated by simple baffles at any point.</u> It comes simply from the fact that the incoming momentum is ordinarily not in line with the outgoing momentum. It will be shown in the body of the report that the incoming air is made to work on an arm $\frac{1}{2}R$ which is quite considerable and that the outgoing air similarly acts on an arm of considerable length, if or when the flow is deflected in a rearward direction. The design referred to above as the NACA method avoids or may avoid the pitching moment with theoretical perfection. In the limit, therefore, a design with zero or nearly zero pitching moment is entirely possible. The magnitude of the pitching moment of the regular arrangement is indicated in the theoretical section of this report.

This report contains two separate main sections. The first covers certain theoretical developments, including some new ones of fundamental aspects. The theory of the line sink in a two-dimensional wing represents a classical solution in the airfoil theory. Closed expressions are given for the hitherto unknown pitching moment. In the second section of this report, the rather extensive experimental literature has been analyzed and consolidated. Finally, there is a short appendix giving certain specific design and performance problems.

ACKNOWLEDGEMENT

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THEORETICAL DEVELOPMENTS

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SECTION I

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LIFTING FAN AIRCRAFT

I. INTRODUCTION

To design an airplane of a nonconventional type characterized by at least partial use of fans to produce lift requires a knowledge of new effects not entering into the design of normal airplanes. Random experiments and flight testing may, at most, show the difficulties, but without an adequate theory such testing rarely leads to more than an accumulation of scattered and sometimes contradictory findings. On the other hand, while the theory may not directly provide complete answers, it is sufficient that it provide a clear understanding of all principles involved and provide means for an orderly solution of well-defined problem areas by experimentation or numerical calculations or both. Only in extremely simple cases may a solution be found without a clear understanding of the theory. The lifting fan aircraft does not fall into this category as experience has shown.

We shall give some elementary considerations of the problem of the duct or propeller enclosure commonly appearing in fan-lifted aircraft. The purpose is to show the underlying principles which may lead to better understanding of the problems involved and to show means for solving such problems by experiments or calculations.

II. GENERAL CONSIDERATIONS

A. The Effect of a Duct

Surprisingly enough it will be shown that the propeller duct serves only one useful purpose, namely to reduce the diameter of the propeller. It appears that all other effects are unfavorable and must be recognized as such.

Let us consider a propeller or fan in an opening in an infinite wall (see Figure II-1). Let the volume of air be of a density ρ and let the volume be Q per unit time. We shall conduct what is called a "Gedanken experiment" by putting a series of baffles as shown in the figure. The cross sectional areas of the channels leading from the hemispherical area down to the circle of the propeller plane are all reduced in the ratio of two at the upper end to one at the lower.

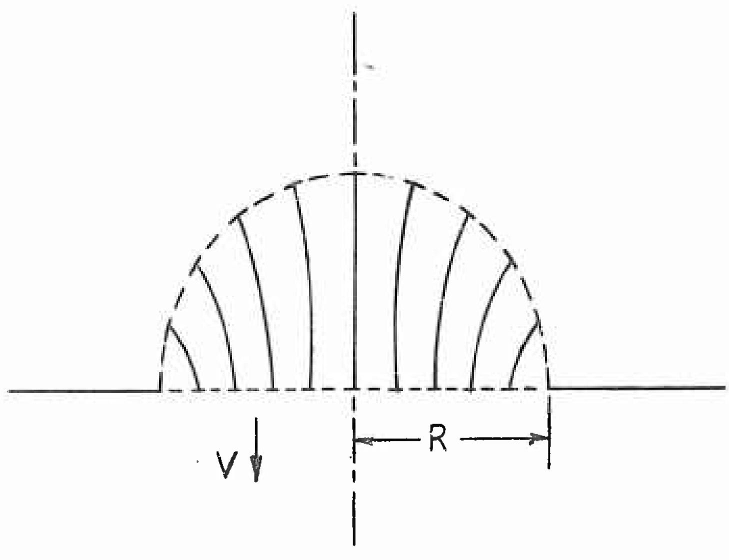


Figure II-1

The channels are frictionless and a constant negative pressure is maintained at the plane of the propeller, or rather, immediately in front of same.

The velocity of the medium in the hemispherical infinity is then

$$V = \frac{Q}{2\pi r^2} \qquad \text{(II-1)}$$

The negative pressure beyond the circular opening of radius R is then:

$$p = -\frac{1}{2}\rho\left[\frac{Q}{2\pi r^2}\right]^2 \qquad \text{(II-2)}$$

and the resulting upward force is then given by the integral

$$F = \frac{1}{2}\rho\left(\frac{Q}{2\pi}\right)^2 \int_R^\infty \frac{1}{r^4} 2\pi r\,dr$$

$$= \frac{Q^2}{8\pi}\rho\int_R^\infty \frac{2\,dr}{r^3} = \frac{\rho Q^2}{8\pi R^2} \qquad \text{(II-3)}$$

With uniform (negative) pressure in the circular opening immediately in front of the propeller there is, of course, uniform velocity since the flow is potential flow with no frictional losses.

The velocity in the plane of the circular opening is, therefore,

$$V_c = \frac{Q}{\pi R^2} \qquad \text{(II-4)}$$

and the corresponding local pressure

$$p = -\frac{1}{2}\rho\left[\frac{Q}{\pi R^2}\right]^2 \tag{II-5}$$

The downward momentum may now be calculated for the opening as

$$(p + \rho V_c^2)\pi R^2 = \left[-\frac{1}{2}\rho\left(\frac{Q}{\pi R^2}\right)^2 + \rho\left(\frac{Q}{\pi R^2}\right)^2\right]\pi R^2$$

$$= \frac{1}{2}\rho\left(\frac{Q}{\pi R^2}\right)^2 \pi R^2 \tag{II-6}$$

$$= \frac{1}{2}\rho\frac{Q^2}{\pi R^2}$$

There is thus an equal and opposite (or upward) force on the wall and the baffles equal to

$$F = \frac{1}{2}\rho\frac{Q^2}{\pi R^2} \tag{II-7}$$

Since, as shown above, $\frac{1}{8}\rho\frac{Q^2}{\pi R^2}$ is carried on the infinite wall beyond the radius R , the remainder, or

$$F_B = \frac{3}{8}\frac{Q^2}{\pi R^2} \tag{II-8}$$

is carried as an upward force on the baffles.

Finally the propeller must restore the pressure back to zero and consequently carries an upward force

$$F_p = p\pi R^2 = \frac{1}{2}\rho\frac{Q^2}{\pi R^2} \tag{II-9}$$

Briefly, therefore, in the ideal case with a propeller located in a "large" surface and provided with "ideal" inlet baffles, the propeller carries exactly one half of the total force.

In fact, it may be shown in general that the force on the propeller situated in an infinite wall with an ideal inlet duct (as in Figure II-2) the propeller, when properly loaded, again carries exactly one half of the total force.

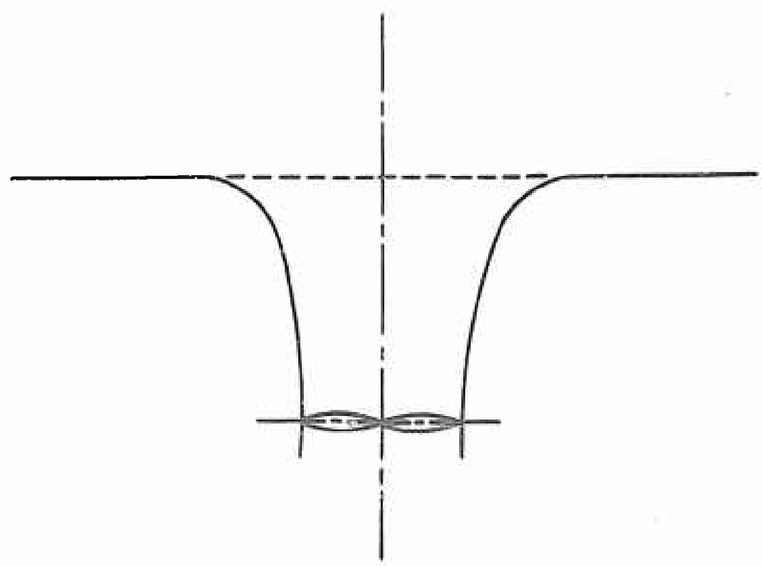


Figure II-2

In brief, a free propeller under ideal conditions operates with a velocity at the plane of the propeller equal to one half of the final velocity in the wake. If the propeller were to operate in the region of the final velocity (as in Figure II-2) - (normally at least one radius behind the plane), then the cross sectional area of the propeller disc would be exactly one half of the normal area and the velocity would be double, in other words, the same horsepower on half the area. To achieve this result it is necessary to install an ideal duct or baffle system. In practice it must be stated that a considerable loss in drag will be caused by such inlet or baffle system.

As final remarks, the condition of an infinite plane can be relaxed without undue loss in lift to, let us say, a plane equal in area to a few times the propeller disc area, but the condition of ideal inlet vanes cannot be relaxed without a large drop in the efficiency of the propeller due to resulting improper velocity distribution, In fact, a serious fault of the most common design is the excessive velocity at the tip of the propeller. The complicating but important effect of forward velocity will be considered in the following article.

B. The Moment Acting on a Surface With a Ducted Fan

There exists a simple and perfectly general equation for the moment of a source (or sink) in a plane. As this problem has a direct bearing on the problem of the lifting fan, we shall show the development in the following. There is also a simple expression for the corresponding ideal "drag". This material is given to show the principles involved.

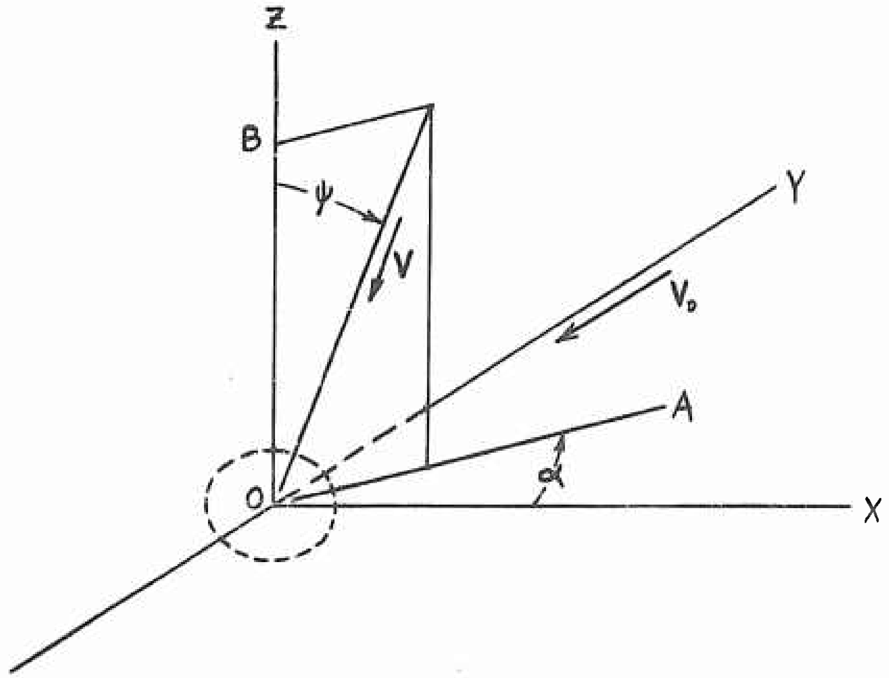


Figure II-3

In Figure II-3 let there be a sink at the origin in the x - y plane. At a distance the resulting velocity due to the sink is directed along the radius vector towards the origin. Let this velocity V be in a plane BOA containing the z - axis and the line OA which forms an angle α

with the x-axis. The velocity vector forms an angle ψ with the vertical or z-axis. One has the following components of the velocity vector

$$\begin{aligned} V_x &= -V\sin\psi\cos\alpha \\ V_Y &= -V\sin\psi\sin\alpha \\ V_z &= -V\cos\psi \end{aligned} \tag{II-10}$$

Further, let the forward velocity be V_o directed as in Figure II-3 along the negative y-axis and uniform in the upper half space.

Let us next draw a hemispherical control surface of radius R around the origin O. We shall now calculate the momentum entering such control surface. This momentum corresponds to the "drag". We shall next calculate the moment of the momentum entering the control surface. This quantity gives the pitching moment resulting from the sink.

An element of the surface is given by

$$d\Omega = r^2 \sin\psi \, d\alpha \, d\psi \tag{II-11}$$

The component of this surface element in the y-direction is

$$d\Omega_Y = d\Omega \sin\psi \sin\alpha = r^2 \sin^2\psi \sin\alpha \, d\psi \, d\alpha \tag{II-12}$$

We shall see that the other components $d\Omega_z$ and $d\Omega_x$ are of no concern since there is a right-left symmetry together with a fore and aft symmetry in the momentum integral.

As we are only concerned with the momentum along the y-axis the arm around the x - axis is always

$$r \cos \psi$$

which is the vertical distance above the x - y plane.

We are concerned with the momentum along the y - axis or the y - momentum. The unit momentum is

$$M = \rho (V_0 + V_Y) \qquad \text{(II-13)}$$

There is thus an excess of momentum in front and a momentum deficiency in the rear quadrants as compared to the mean value. It is simplest to consider the air as containing a momentum vector ρV_0 in the y direction and a momentum vector ρV in the radial inward direction. It is obvious that the second vector does not contribute to the moment of the momentum, and one, therefore, need only consider the main vector

$$M = \rho V_0 \qquad \text{(II-14)}$$

and calculate the quantity of air crossing the control surface. This quantity is simply

$$V d\Omega$$

since the constant velocity V_0 contributes nothing.

With

$$d\Omega = r^2 \sin \psi \, d\alpha \, d\psi \qquad \text{(II-15)}$$

and the arm

$$a = r\cos\psi \tag{II-16}$$

There remains for one quadrant

$$\frac{1}{4}M = \int_{\psi=0}^{\frac{\pi}{2}} \int_{\alpha=0}^{\frac{\pi}{2}} \rho V V_0 r^2 \sin\psi \, d\alpha \, d\psi \cdot r\cos\psi$$

$$= \rho V V_0 r^3 \frac{\pi}{4}$$

or

$$M = \rho V V_0 \cdot \pi R^3 \tag{II-17}$$

where R is the radius of the control surface.

This is the significant expression for the pitching moment. Since the volume per second through the sink is

$$Q = 2\pi R^2 V \tag{II-18}$$

one has also

$$M = \rho V_0 Q \cdot \frac{R}{2} \tag{II-19}$$

Further, as the "lift" is

$$L = \rho Q V_1 \tag{II-20}$$

where V_1 is the velocity of the jet, one has further for the pitching moment

$$M = L \cdot \frac{V_0}{V_1} \cdot \frac{1}{2} R \tag{II-21}$$

It is thus shown that the pitching moment increases linearly with the "forward" speed V_0 . The quantity R or the radius of the control surface is related to the dimensions of the "airfoil" or surface. It can be shown, however, that this is a general relation. A large surface around a duct is therefore the cause of a large pitching moment.

By a simple integration of the moment one obtains for the "drag"

$$D = \rho V_0 Q \tag{II-22}$$

and consequently

$$\frac{L}{D} = \frac{V_1}{V_0} \tag{II-23}$$

where V_0 again is the forward velocity and V_1 the jet velocity.

The "arrested" momentum or "drag" $\rho Q V_0$ thus acts on an arm $\frac{1}{2}R$ above the lifting surface or the propeller plane where R is the "mean" radius of such plane or area.

C. Moment of Momentum Due to Sink in the Center of a Rectangular Plate

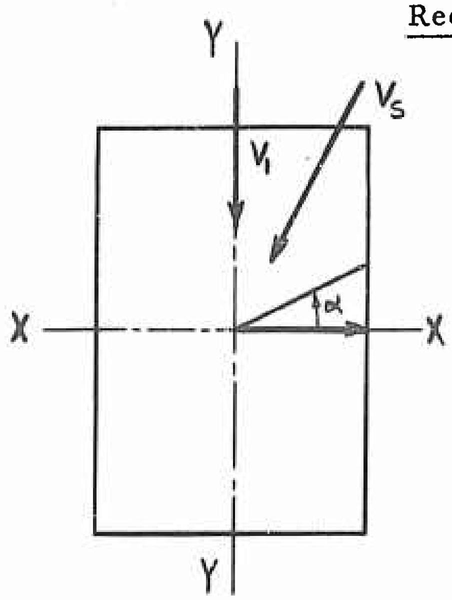


Figure II-4

Let V_1 be the constant velocity in the negative Y direction and V_S the velocity due to the sink in the center of the rectangular plate. We shall initially consider the velocity V_S to be given by the formula

$$V_S = \frac{Q}{2\pi r^2} \tag{II-24}$$

which is equivalent to extending the plate to infinity in all directions.

Q is the volume of air per unit of time and r the radial distance from the center. We shall show, in a later article, how the case of a finite plate may be obtained. The purpose of the present development is to show the general effect of the orientation of the rectangular surface on the moment of the momentum caused by the sink.

The pressure is given by

$$P = P_o - \frac{1}{2}\rho(\bar{V}_1 + \bar{V}_S)^2$$

$$= P_o - \frac{1}{2}\rho(\bar{V}_1^2 + \bar{V}_S^2) - \rho\bar{V}_1 \cdot \bar{V}_S \tag{II-25}$$

Since the two first terms are symmetric with respect to the center of the plate it is only necessary to consider the term

$$P = -\rho \bar{V}_1 \cdot \bar{V}_S = -\rho V_1 V_S \sin\alpha \tag{II-26}$$

where α is the angle of the radius vector r with the X axis as the only term contributing to the moment around the X axis. As each quadrant in Figure (II-4) contributes the same amount we may write for the moment around the X axis

$$M = 4\int_0^{Y_1}\int_0^{X_1} \rho V_1 V_S \sin\alpha \cdot Y dX dY \tag{II-27}$$

By reference to Figure(II-1) it may be verified that this is a "nose-up" pitching moment as expected.

With $V_S = \dfrac{Q}{2\pi r^2}$ and $\sin\alpha = \dfrac{Y}{r}$ one has

$$M = 4\rho V_1 \frac{Q}{2\pi}\int_0^{Y_1}\int_0^{X_1} \frac{Y^2}{r^3} dX dY \tag{II-28}$$

We shall obtain the double integral

$$\int_0^{Y_1} Y^2 dY \int_0^{X_1} \frac{dX}{\sqrt{X^2+Y^2}^{\,3}} = \int_0^{Y_1} Y^2 dY \left[\frac{X}{Y^2\sqrt{X^2+Y^2}}\right]_0^{X_1}$$

$$= X_1 \int_0^{Y_1} \frac{dY}{\sqrt{X_1^2+Y^2}}$$

$$= X_1 \sinh^{-1}\frac{Y_1}{X_1}$$

and the moment may be written:

$$M = \frac{2}{\pi} \rho V_1 Q X_1 \operatorname{Sinh}^{-1} \frac{Y_1}{X_1} \tag{II-29}$$

Also in the form

$$M = \rho V_1 Q \frac{2}{\pi} X_1 \operatorname{Sinh}^{-1} \frac{Y_1}{X_1} \tag{II-30}$$

it is seen that the arm a is

$$a = \frac{2}{\pi} X_1 \operatorname{Sinh}^{-1} \frac{Y_1}{X_1} \tag{II-31}$$

Let us consider a rectangle of a given area $x_1 y_1 = 1$ or the area $A = 4$. One may then show the effect of the orientation of such rectangular area by the following table for a moment of the momentum or the pitching moment M or rather the arm a, which is really a moment coefficient.

TABLE II-1

$\frac{Y_1}{X_1}$	1/2	1	2	4
X_1	$\sqrt{2}$	1	$\frac{1}{\sqrt{2}}$	1/2
$\operatorname{Sinh}^{-1}\frac{Y_1}{X_1}$	0.48	.88	1.44	2.10
a	.44	.56	.62	.67

Note that a change in the "aspect ratio" in the ratio of 1:8 only changes the resulting moment by fifty percent and that the fore and aft orientation thus is slightly more objectionable as far as the moment is concerned. Note that the moment on a square, as expected, is almost exactly equal to that of a circle of the same area. These effects are all, of course, caused by the fact that the pressures decrease as the square of the distance from the origin. These effects may be given more precisely by a method to be given in a subsequent communication. The purpose of the present article is to show that for a given area the moment effect at a central sink is greater when the "aspect ratio" of the surface is small, that is, if the surface is oriented in a fore and aft direction.

III. TWO-DIMENSIONAL WING THEORY

A. Wing With Line Sink on Upper Surface and Discrete Line Jet on the Lower Surface

In the following is given the theory of a wing with jets showing the integrated effect of such jets on the lift and the moment of the wing.

A source at (x_1 , y_1) of strength $\mathcal{E}$ and another source at $(x_1 , -y_1)$ of the same strength gives a symmetric function (about the X axis) (see Figure III-1)

$$\phi_s = \frac{\mathcal{E}}{4\pi}\left\{\log\left[(X-X_1)^2+(Y-Y_1)^2\right]+\log\left[(X-X_1)^2+(Y+Y_1)^2\right]\right\} \qquad \text{(III-1)}$$

If the points (x_1 , y_1) and $(x_1 , -y_1)$ are on a circle of unit radius, the quantities

$$M = \sqrt{(X-X_1)^2+(Y-Y_1)^2}$$

and

$$N = \sqrt{(X-X_1)^2+(Y+Y_1)^2}$$

may be expressed in angular coordinates α

$$M^2 = 2\left[1-\cos(\alpha-\alpha_1)\right]$$

$$N^2 = 2\left[1-\cos(\alpha+\alpha_1)\right]$$

and one has in angular coordinates

$$\phi_s = \frac{\varepsilon}{2\pi}(\log M + \log N) \tag{III-2}$$

UNIT CIRCLE SHOWING COORDINATES USED

Throughout Chapter III the reference length (or unit) corresponds to the radius of the circle or the half chord of the wing.

$\cos\alpha = -X$

$\sin\alpha = Y$

$\tan\frac{\alpha_1}{2} = \sqrt{\frac{1+X_1}{1-X_1}}$

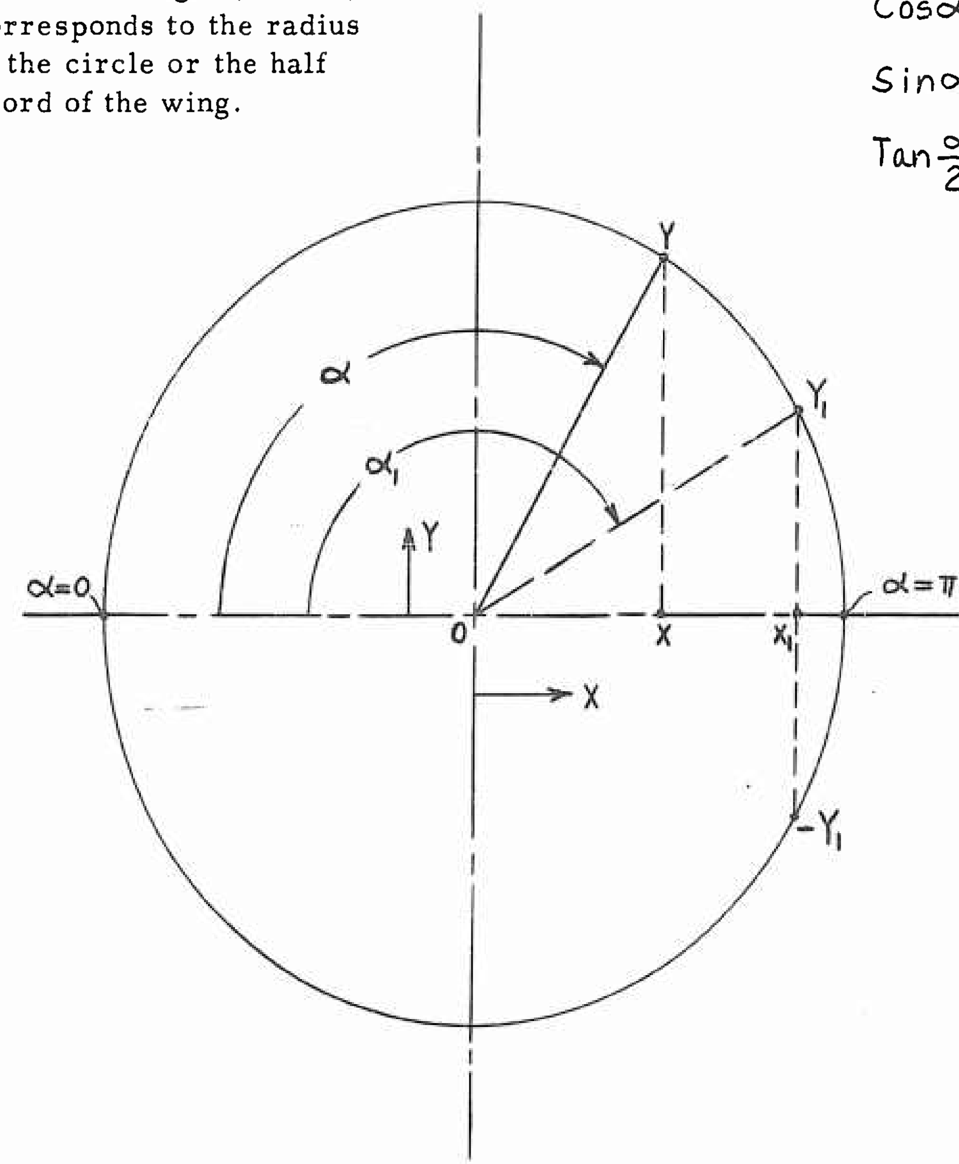


Figure III-1

The velocity at the surface of the circle is

$$V_S = \frac{d\phi}{d\alpha} = \frac{\varepsilon}{2\pi}\left(\frac{1}{M}\frac{dM}{d\alpha} + \frac{1}{N}\frac{dN}{d\alpha}\right) \tag{III-3}$$

With the circle transformed into a straight line one has for the velocity along this line with a source of strength ε on each side

$$V_S = 2\frac{d\phi}{dX} = 2\frac{d\phi}{d\alpha}\cdot\frac{d\alpha}{dX} = \frac{\varepsilon}{\pi}\left[\frac{1}{M}\frac{dM}{d\alpha} + \frac{1}{N}\frac{dN}{d\alpha}\right]\frac{d\alpha}{dX} \tag{III-4}$$

One has further

$$\begin{cases} \dfrac{2}{M}\dfrac{dM}{d\alpha} = \dfrac{\sin(\alpha-\alpha_1)}{1-\cos(\alpha-\alpha_1)} = \cot\dfrac{\alpha-\alpha_1}{2} \\[2ex] \dfrac{2}{N}\dfrac{dN}{d\alpha} = \dfrac{\sin(\alpha+\alpha_1)}{1-\cos(\alpha+\alpha_1)} = \cot\dfrac{\alpha+\alpha_1}{2} \end{cases}$$

and

$$\cot\frac{\alpha-\alpha_1}{2} + \cot\frac{\alpha+\alpha_1}{2} = -\frac{2\sin\alpha}{\cos\alpha - \cos\alpha_1}$$

Also $\dfrac{d\alpha}{dX} = \dfrac{1}{\sin\alpha}$ and one has finally

$$V_S = 2\frac{d\phi}{dX} = -\frac{\varepsilon}{\pi}\,\frac{\sin\alpha}{\cos\alpha - \cos\alpha_1}\cdot\frac{1}{\sin\alpha}$$

$$V_S = -\frac{\varepsilon}{\pi}\cdot\frac{1}{\cos\alpha - \cos\alpha_1} \tag{III-5}$$

Note that because of the symmetry one obtains with a source ε on each side actually a simple source of strength 2ε in two-dimensional space. The line between $x = -1$ and $x = 1$ has no influence in the flow as expected. With x - coordinates the equation then reads

$$V_s = \frac{\varepsilon}{\pi} \cdot \frac{1}{x - x_1}$$

We shall also need a non-symmetrical flow configuration with a source at (x_1, y_1) but with a sink at $(x_1, -y_1)$. The development is entirely similar with the exception that the quantity N now has a negative sign.

There results

$$\cot\frac{\alpha-\alpha_1}{2} - \cot\frac{\alpha+\alpha_1}{2} = -\frac{2\sin\alpha_1}{\cos\alpha - \cos\alpha_1}$$

and for the non-symmetrical velocity

$$V_n = -\frac{\varepsilon}{\pi} \frac{\sin\alpha_1}{\cos\alpha - \cos\alpha_1} \cdot \frac{1}{\sin\alpha} \qquad \text{(III-6)}$$

This expression may be written

$$V_n = \frac{\varepsilon}{\pi} \cdot \frac{1}{x - x_1} \frac{\sqrt{1 - x_1^2}}{\sqrt{1 - x^2}}$$

Note that the velocity V_n goes to infinity at $x = \pm 1$ and at $x = x_1$ as it should.

We shall next calculate the flow function for a sink of strength $\mathcal{E}$ designated as $-\mathcal{E}$ located on the upper surface only. It is observed that if one adds the symmetric and the non-symmetric flow functions already developed one obtains the effect of a double source on the upper surface, since the source and the sink cancel on the lower surface. We may, therefore, write for the velocity resulting from a sink ($-\mathcal{E}$) only on the upper surface:

$$V = -\frac{1}{2}(V_s + V_n)$$

$$V = \frac{\mathcal{E}}{2\pi}\left(\frac{1}{\cos\alpha - \cos\alpha_1} + \frac{1}{\cos\alpha - \cos\alpha_1} \cdot \frac{\sin\alpha_1}{\sin\alpha}\right)$$

To avoid the infinity in the velocity at $x = 1$ or $\alpha = \pi$ or to comply with the Kutta condition, one must introduce a circulation function

$$\phi = \frac{\Gamma}{2\pi}\alpha \tag{III-7}$$

where Γ is the clockwise circulation constant.

The velocity due to circulation is then

$$V_\Gamma = \frac{d\phi}{dX} = \frac{d\phi}{d\alpha} \cdot \frac{d\alpha}{dX} = \frac{\Gamma}{2\pi} \cdot \frac{1}{\sin\alpha} \tag{III-8}$$

Now with a sink on the upper surface

$$V_n + V_\Gamma = \text{finite at } \alpha = \pi$$

$$\frac{\varepsilon}{2\pi} \cdot \frac{1}{-1-\cos\alpha_1} \cdot \frac{\sin\alpha_1}{0} + \frac{\Gamma}{2\pi} \cdot \frac{1}{0} = \text{finite}$$

or

$$\Gamma = \varepsilon \frac{\sin\alpha_1}{1+\cos\alpha} = \varepsilon \tan\frac{\alpha_1}{2} = \varepsilon \frac{Y_1}{1-X_1} \tag{III-9}$$

Hence the induced circulation goes to infinity for $\alpha_1 = \pi$ or when the sink is near the trailing edge. For the sink in the middle $\alpha_1 = \frac{\pi}{2}$, $\Gamma = \varepsilon$ and for the sink near the leading edge $\Gamma \to 0$.

B. <u>The Effect of the Combined Flow Field</u>

There are now altogether four velocities V_s, V_n, V_Γ and a superimposed constant velocity W. We shall rewrite these quantities,

$$V_s = \frac{\varepsilon}{2\pi} \cdot \frac{1}{\cos\alpha + X_1}$$

$$V_n = \frac{\varepsilon}{2\pi} \cdot \frac{Y_1}{\cos\alpha + X_1} \cdot \frac{1}{\sin\alpha} \tag{III-9a}$$

$$V_\Gamma = \frac{\varepsilon}{2\pi} \cdot \frac{Y_1}{1-X_1} \cdot \frac{1}{\sin\alpha}$$

$$W = \frac{\varepsilon}{2\pi} \cdot W_o$$

(where $W_o = \frac{2\pi}{\varepsilon} W$ for symmetry)

There is thus a symmetric component comprising

$$W + V_s$$

and a non-symmetric component

$$V_n + V_\Gamma$$

The acting negative pressure on the upper surface is then

$$\frac{1}{2}\rho\left[W + V_s + V_n + V_\Gamma\right]^2$$

and on the lower surface

Similarly

$$\frac{1}{2}\cdot\rho\left[W + V_s - (V_n + V_\Gamma)\right]^2$$

The net lift force therefore is simply the difference

$$\Delta L = 2\rho(W + V_s)(V_n + V_\Gamma) \tag{III-10}$$

and the integrated lift force is then

$$L = 2\rho\int_{-1}^{1}(W + V_s)(V_n + V_\Gamma)dX \tag{III-11}$$

and the clockwise moment about the point x_1 or the location of the sink is thus

$$M_{X_1} = 2\rho\int_{-1}^{1}(W + V_s)(V_n + V_\Gamma)(X - X_1)dX \tag{III-12}$$

Inserting the expressions for the velocities the integral becomes

$$L = 2\rho\left(\frac{\varepsilon}{2\pi}\right)^2 \int_0^{+\pi} \left(W_0 + \frac{1}{\cos\alpha + X_1}\right)\left[\frac{1}{\cos\alpha + X_1} + \frac{1}{1 - X_1}\right]\frac{Y_1}{\sin\alpha}\,\sin\alpha\, d\alpha$$

$$= 2\rho Y_1\left(\frac{\varepsilon}{2\pi}\right)^2 \int_0^{+\pi} \left(W_0 + \frac{1}{\cos\alpha + X_1}\right)\left(\frac{1}{\cos\alpha + X_1} + \frac{1}{1 - X_1}\right) d\alpha$$

or rearranged

$$= 2\rho Y_1\left(\frac{\varepsilon}{2\pi}\right)^2 \int_0^{+\pi} \left\{\left(W_0 + \frac{1}{1 - X_1}\right)\frac{1}{\cos\alpha + X_1} + \frac{1}{(\cos\alpha + X_1)^2} + W_0\frac{1}{1 - X_1}\right\} d\alpha$$

(III-13)

The first term contributes nothing.

The third term contributes

$$L_1 = 2\rho Y_1\left(\frac{\varepsilon}{2\pi}\right)^2 W_0 \cdot \frac{1}{1 - X_1}\pi = \rho\,\varepsilon\, W\left(\frac{Y_1}{1 - X_1}\right) \qquad \text{(III-14)}$$

For the second term we must evaluate the integral

$$\int_0^{+\pi} \frac{d\alpha}{(\cos\alpha + X_1)^2} = \frac{1}{1 - X_1^2} \cdot \left.\frac{\sin\alpha}{X_1 + \cos\alpha}\right]_0^{+\pi}$$

There is a singularity at x_1 . We must, therefore, omit the range $x_1 - \Delta x_1$ to $x_1 + \Delta x_1$. We get, therefore:

$$\frac{1}{1-X_1^2} \cdot \frac{2Y_1}{\Delta X_1} = \frac{2}{Y_1 \Delta X}$$

The contribution to the integral is thus

$$L_2 = 2\rho\left(\frac{\varepsilon}{2\pi}\right)^2 Y_1 \cdot \frac{2}{Y_1 \Delta X} = \rho\left(\frac{\varepsilon}{\pi}\right)^2 \cdot \frac{1}{\Delta X} \tag{III-15}$$

This lift is carried on the wing beyond (outside) the opening $2\,\Delta_x$.

C. Effect of Propeller and Baffle System

With a perfect baffle system which may be approached in a practical case there is no loss at the entrance to the "ducted" propeller, and the ideal case potential flow is, in any case, the most important as the limiting case.

The pressure difference produced by the propeller is equal to $\frac{1}{2}\rho V^2$ where V is the velocity at the propeller or in the propeller plane. This pressure differential will exactly restore the pressure to normal. The thrust of the "propeller" is thus $\frac{1}{2}\rho V^2 \cdot 2\Delta X$ where $2\Delta X$ corresponds to the opening or

$$T_p = \rho V^2 \Delta X = \rho \left(\frac{\varepsilon}{2\Delta X}\right)^2 \Delta X = \rho \frac{\varepsilon^2}{4\Delta X} \tag{III-16}$$

Note that the propeller does not carry the full thrust. The total thrust caused by the sink equal mass per second $\rho\varepsilon$ multiplied by the final velocity $\frac{\varepsilon}{2\Delta X}$ or the total thrust

$$T_T = \rho \frac{\varepsilon^2}{2\Delta X} \tag{III-17}$$

Thus exactly one half of the total thrust is carried by the propeller in the duct. The remainder is carried by the "perfect" baffle system except for the relatively small portion (Equation (III-15) :

$$\frac{1}{\pi^2} \rho \frac{\varepsilon^2}{\Delta X}$$

carried by the remainder of the wing beyond the opening $2\Delta X$, and the independent contribution of the induced circulation shown above.

There are thus the following contributions to the lift:

The contribution due to induced circulation

$$L_1 = \rho \frac{Y_1}{1 - X_1} \varepsilon W \tag{III-18}$$

and the contribution directly due to the sink with the thrust carried one half by the propeller and one half by the baffle system and the wing, in total

$$L_2 = \rho \frac{\varepsilon^2}{2\Delta X} = \rho \varepsilon W_D \qquad \text{(III-19)}$$

where W_D there is the velocity in the plane of the propeller or the total downward velocity $\dfrac{\varepsilon}{2\Delta X}$.

D. Pitching Moment

We shall next proceed to evaluate the moment around the point x_1 , the location of the sink.

$$M = 2\rho\left(\frac{\varepsilon}{2\pi}\right)^2 Y_1 \int_0^{+\pi} \left(W_0 + \frac{1}{\cos\alpha + X_1}\right)\left(\frac{1}{\cos\alpha + X_1} + \frac{1}{1 - X_1}\right)(\cos\alpha + X_1)\,d\alpha$$

$$= 2\rho\left(\frac{\varepsilon}{2\pi}\right)^2 Y_1 \int_0^{+\pi} \left\{\left(W_0 + \frac{1}{1 - X_1}\right) + \frac{1}{\cos\alpha + X_1} + \frac{W_0}{1 - X_1}(\cos\alpha + X_1)\right\} d\alpha$$

$$= 2\rho\left(\frac{\varepsilon}{2\pi}\right)^2 Y_1 \int_0^{+\pi} \frac{W_0 + 1}{1 - X_1}\,d\alpha = 2\rho\left(\frac{\varepsilon}{2\pi}\right)^2 Y_1 \cdot \frac{W_0 + 1}{1 - X_1}\,\pi$$

or finally

$$M = \rho \varepsilon \frac{Y_1}{1-X_1}\left(W + \frac{\varepsilon}{2\pi}\right) \tag{III-20}$$

We shall next indicate the length of the moment arm. With the moment in dimensional form, with b as the half chord

$$M = \rho \varepsilon b \frac{Y_1}{1-X_1}\left(W + \frac{\varepsilon}{2\pi b}\right)$$

and the force or lift

$$L = \rho \varepsilon \left(\frac{Y_1}{1-X_1} W + W_D\right)$$

one has for the arm

$$a = \frac{M}{L} = \frac{b \frac{Y_1}{1-X_1}\left(W + \frac{\varepsilon}{2\pi b}\right)}{\frac{Y_1}{1-X_1} W + W_D}$$

or also

$$a = \frac{b \frac{W}{W_D} + \frac{\varepsilon}{2\pi W_D}}{\frac{W}{W_D} + \frac{1-X_1}{Y_1}} \tag{III-21}$$

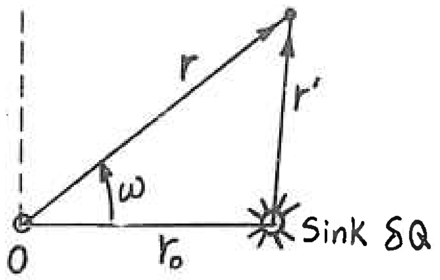
The arm a is given in terms of the half chord as unity. The ratio W/W_D is the ratio of the forward velocity in terms of the downward velocity of the propeller $W_D = \frac{\varepsilon}{2\Delta X}$.

These results are all given for zero angle of attack. The effect of the angle of attack may be directly superimposed. This problem will be taken up in a later section of this report. There is a ground effect and also a displacement effect of a (large) jet on the lower wing surface which are not treated at present.

IV. FLOW FIELD FOR UNIFORM SINK DISTRIBUTION IN A CIRCULAR DISK

A. Point Sink

The velocity potential for the flow of a point sink of strength δQ located at a distance r_o from the origin is given by



$$\delta\phi(r,\omega) = \frac{\delta Q}{4\pi r'} \tag{IV-1}$$

$$\delta\phi(r,\omega) = \frac{\delta Q}{4\pi r}\sum_{n=0}^{\infty}\left(\frac{r_o}{r}\right)^n P_n(\cos\omega) \qquad \text{for } \frac{r}{r_o} > 1$$

$$\delta\phi(r,\omega) = \frac{\delta Q}{4\pi r_o}\sum_{n=0}^{\infty}\left(\frac{r}{r_o}\right)^n P_n(\cos\omega) \qquad \text{for } \frac{r}{r_o} < 1$$

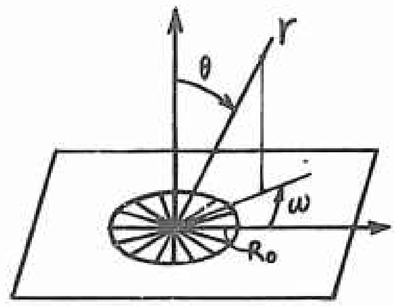
For an angle $\omega = 90^\circ$ the above Equation (IV-1) becomes

$$\delta\phi(r,\omega)_{\omega=90^\circ} = \frac{\delta Q}{4\pi r}\sum_{n=0}^{\infty}\left(\frac{r_o}{r}\right)^n P_n(0) \qquad \text{for } \frac{r}{r_o} > 1$$

$$\delta\phi(r,\omega)_{\omega=90^\circ} = \frac{\delta Q}{4\pi r_o}\sum_{n=0}^{\infty}\left(\frac{r}{r_o}\right)^n P_n(0) \qquad \text{for } \frac{r}{r_o} < 1 \tag{IV-2}$$

B. Circular Disk Sink

The velocity potential along the axis of symmetry of sinks uniformly distributed over a circular disk is found by integrating Equation (VI-2) over the area of the circular disk.



1. Case (a): $r \geq R_0$

For $r > R_0$ i.e. $\frac{r}{r_0} > 1$ the velocity potential along the axis $\theta = 0$ becomes

$$\phi(r,\theta)_{\substack{\theta=0\\ \omega=90^\circ}} = \int_0^{R_0}\int_0^{2\pi} \frac{\delta Q}{4\pi r} \sum_{n=0}^{\infty} \left(\frac{r_0}{r}\right)^n P_n(0)\, r_0\, dr_0\, d\varphi \tag{IV-3}$$

Integrating we get

$$\phi(r,0) = \frac{\delta Q\, \pi R_0^2}{2\pi r} \sum_{n=0}^{\infty} \frac{P_n(0)}{n+2} \left(\frac{R_0}{r}\right)^n$$

Substituting for $\delta Q \cdot \pi R_0^2 = Q$ we find

$$\phi(r,0) = \frac{Q}{2\pi} \sum_{n=0}^{\infty} \frac{P_n(0)}{n+2} R_0^n \frac{1}{r^{n+1}} \tag{IV-4}$$

Since the flow field is axially symmetric with respect to the axis $\theta = 0$ we know that the Laplace equation

$$\nabla^2\phi(r,\theta) = \frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) = 0 \qquad \text{(IV-5)}$$

has the following non-singular general solutions

$$\phi(r,\theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta) + \sum_{n=0}^{\infty} B_n \frac{1}{r^{n+1}} P_n(\cos\theta) \qquad \text{(IV-6)}$$

For $\theta = 0$ the above equation becomes

$$\phi(r,0) = \sum_{n=0}^{\infty} A_n r^n P_n(1) + \sum_{n=0}^{\infty} B_n \frac{1}{r^{n+1}} P_n(1) \qquad \text{(IV-7)}$$

Comparing Equations (IV-4) and (IV-7) we get

$$A_n \equiv 0 \qquad \& \qquad B_n = \frac{Q}{2\pi}\frac{P_n(0)}{n+2}R_0^n \qquad \text{(IV-8)}$$

Consequently for $r > R_0$ the velocity potential $\phi(r,\theta)$ is given by

$$\phi(r,\theta) = \frac{Q}{2\pi}\sum_{n=0}^{\infty} R_0^n \frac{P_n(0)}{n+2}\frac{1}{r^{n+1}} P_n(\cos\theta)$$

or

$$\phi(r,\theta) = \frac{Q}{2\pi r}\sum_{n=0}^{\infty}\left(\frac{R_0}{r}\right)^n \frac{P_n(0)\,P_n(\cos\theta)}{n+2} \qquad \text{(IV-9)}$$

2. Case (b): $r \leq R_0$

For $r < R_0$ we have that

$$\frac{r}{r_0} > 1 \quad \text{for} \quad 0 < r_0 < r$$

and

$$\frac{r}{r_0} < 1 \quad \text{for} \quad r < r_0 < R_0 \qquad \text{(IV-10)}$$

Thus the velocity potential $\phi(r,0)$ at the axis $\theta = 0$ is found by integrating Equations (IV-2) from 0 to r and from r to R respectively, i.e.

$$\phi(r,0) = \int_0^r \int_0^{2\pi} \frac{\delta Q}{4\pi r} \sum_{n=0}^{\infty} \left(\frac{r_0}{r}\right)^n P_n(0)\, r_0\, dr_0\, d\varphi$$
$$+ \int_r^{R_0} \int_0^{2\pi} \frac{\delta Q}{4\pi r_0} \sum_{n=0}^{\infty} \left(\frac{r}{r_0}\right)^n P_n(0)\, r_0\, dr_0\, d\varphi \qquad \text{(IV-11)}$$

Performing the integration we get

$$\phi(r,0) = \frac{\delta Q}{2} \sum_{n=0}^{\infty} P_n(0) \left\{ \frac{1}{r^{n+1}} \int_0^r r_0^{n+1} dr_0 + r^n \int_r^{R_0} \frac{dr_0}{r_0^n} \right\}$$

or

$$\phi(r,0) = \frac{\delta Q}{2} \sum_{\substack{n=0 \\ n \neq 1}}^{\infty} P_n(0) \frac{r}{n-1} \left\{ \frac{2n+1}{n+2} - \left(\frac{r}{R_0}\right)^{n-1} \right\}$$

Substituting $\pi R_0^2 \cdot \delta Q = Q$ we find

$$\phi(r,0) = \frac{Qr}{2\pi R_0^2} \sum_{\substack{n=0 \\ n\neq 1}}^{\infty} \frac{P_n(0)}{n-1}\left[\frac{2n+1}{n+2} - \left(\frac{r}{R_0}\right)^{n-1}\right] \qquad \text{(IV-12)}$$

Making use of Equation (IV-6), since the flow field is axially symmetric, and comparing Equation (IV-7) we get

$$A_1 = \frac{Q}{2\pi R_0^2} \sum_{\substack{K=0 \\ K\neq 1}}^{\infty} \frac{P_K(0)}{K-1} \frac{2K+1}{K+2}$$

$$A_n = -\frac{Q}{2\pi R_0^{n+1}} \cdot \frac{P_n(0)}{n-1} \qquad \text{for } n \neq 1 \qquad \text{(IV-13)}$$

$$B_n \equiv 0$$

The summation of the first term of Equation (IV-13) is

$$S = \sum_{\substack{K=0 \\ K\neq 1}}^{\infty} \frac{P_K(0)}{K-1} \frac{2K+1}{K+2} \qquad \text{(IV-14)}$$

Since the odd Legendre polynomials $P_{2K+1}(0) \equiv 0$, the above equation might be written as follows

$$S = \sum_{K=0}^{\infty} \frac{4K+1}{(2K-1)(2K+2)} P_{2K}(0)$$

Expanding the coefficient of $P_{2K}(0)$ we get

$$S = \sum_{K=0}^{\infty} \left(\frac{1}{2K-1} + \frac{1}{2K+2}\right) P_{2K}(0) \qquad \text{(IV-15)}$$

From the expansion of the Legendre polynomial at zero we have that

$$P_{2K}(0) = (-1)^K \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2K-1)}{2 \cdot 4 \cdot 6 \cdot 8 \cdots 2K}$$

Consequently, we might write

$$P_{2K+2}(0) = -P_{2K}(0)\frac{2K+1}{2K+2} \qquad \text{(IV-16)}$$

The first sum of Equation (IV-15) might be written as

$$\sum_{K=0}^{\infty} \frac{1}{2K-1} P_{2K}(0) = \sum_{K=1}^{\infty} \frac{P_{2K}(0)}{2K-1} - 1 \qquad \text{(IV-17)}$$

Changing the subscript k to n, where n = k - 1, Equation (IV-17)

becomes

$$\sum_{K=0}^{\infty} \frac{P_{2K}(0)}{2K-1} = \sum_{n=0}^{\infty} \frac{P_{2n+2}(0)}{2n+1} - 1$$

Making use of Equation (IV-16) the above equation might be written as

$$\sum_{K=0}^{\infty} \frac{P_{2K}(0)}{2K-1} = \sum_{n=0}^{\infty} \frac{-P_{2n}(0)}{2n+1}\,\frac{2n+1}{2n+2} - 1$$

or

$$\sum_{K=0}^{\infty} \frac{P_{2K}(0)}{2K-1} + \sum_{n=0}^{\infty} \frac{P_{2n}(0)}{2n+2} = -1$$

Thus we have that Equation (IV-15) becomes

$$S = \sum_{n=0}^{\infty} \left(\frac{1}{2n-1} + \frac{1}{2n+2} \right) P_{2n}(0) = -1$$

or

$$S = \sum_{\substack{K=0 \\ K \neq 1}}^{\infty} \frac{2K+1}{(K-1)(K+2)} P_K(0) = -1 \qquad \text{(IV-18)}$$

And

$$\sum_{\substack{K=0 \\ K \neq 1}}^{\infty} \frac{2K+1}{2(K-1)(K+2)} P_K(0) = -\frac{1}{2} \qquad \text{(IV-19)}$$

Making use of Equations (IV-13) and (IV-18) the velocity potential $\phi(r,\theta)$ becomes

$$\phi(r,\theta) = -\frac{Q}{2\pi R_0} \left\{ \frac{r}{R_0} P_1(\cos\theta) + \sum_{n=0}^{\infty}{}^{*} \frac{P_n(0)}{n-1} \left(\frac{r}{R_0} \right)^n P_n(\cos\theta) \right\} \qquad \text{(IV-20)}$$

where

$$\sum_{K=0}^{\infty}{}^{*} \equiv \sum_{\substack{K=0 \\ K \neq 1}}^{\infty} \quad \& \quad \sum_{n=0}^{\infty}{}^{*} \equiv \sum_{\substack{n=0 \\ n \neq 1}}^{\infty}$$

C. Streamfunction

A streamfunction $\psi(r,\theta)$ is defined from the continuity equation

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 u_r \right) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \cdot u_\theta \right) = 0$$

or

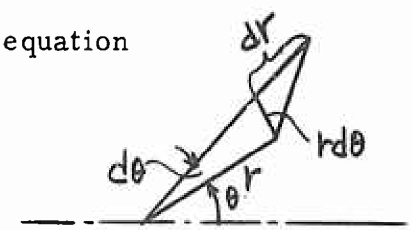
$$\frac{\partial}{\partial r}\left(2\pi r^2 \sin\theta\, u_r\right) + \frac{\partial}{\partial \theta}\left(2\pi r \sin\theta \cdot u_\theta\right) = 0$$

Thus we have:

$$\frac{\partial \psi}{\partial r} = -2\pi r u_\theta \cdot \sin\theta$$

$$\frac{\partial \psi}{\partial \theta} = 2\pi r^2 u_r \cdot \sin\theta \qquad \text{(IV-21)}$$

This definition also satisfies the equation



$$d(\text{Flux}) = d\psi$$

$$d\psi = \frac{\partial \psi}{\partial r} dr + \frac{\partial \psi}{\partial \theta} d\theta \qquad \text{(IV-22)}$$

$$d(\text{Flux}) = 2\pi r^2 \sin\theta \cdot u_r\, d\theta - 2\pi r \sin\theta \cdot u_\theta\, dr$$

1. Case (a): $r \geq R_0$

From Equation (IV-21) we have that

$$u_r = \frac{\partial \phi}{\partial r} = -\frac{Q}{2\pi r^2} \sum_{n=0}^{\infty} \frac{n+1}{n+2} \left(\frac{R_0}{r}\right)^n P_n(0)\, P_n(\cos\theta)$$

$$\text{(IV-23)}$$

$$u_\theta = \frac{\partial \phi}{\partial \theta} = -\frac{Q}{2\pi r^2} \sin\theta \sum_{n=0}^{\infty} \left(\frac{R_0}{r}\right)^n \frac{P_n(0)}{n+2} \frac{dP_n}{d\mu}$$

$$\mu \equiv \cos\theta$$

Thus the first of Equation (IV-21) becomes

$$\frac{\partial\psi}{\partial r} = -2\pi r u_\theta \sin\theta = \frac{Q\sin^2\theta}{r}\sum_{n=0}^{\infty}\left(\frac{R_0}{r}\right)^n \frac{P_n(0)}{n+2}\frac{dP_n}{d\mu} \tag{IV-24}$$

Integrating we get

$$\psi = -Q\sin^2\theta\sum_{n=1}^{\infty}\left(\frac{R_0}{r}\right)^n \frac{P_n(0)}{n(n+2)}\frac{dP_n}{d\mu} + f(\theta) \tag{IV-25}$$

Differentiating Equation (IV-25) with respect to θ we get

$$\frac{\partial\psi}{\partial\theta} = -Q\sum_{n=1}^{\infty}\left(\frac{R_0}{r}\right)^n \frac{P_n(0)}{n(n+2)}\left\{2\sin\theta\cos\theta\frac{dP_n}{d\mu} - \sin^3\theta\frac{d^2P_n}{d\mu^2}\right\} + f'(\theta)$$

or

$$\frac{\partial\psi}{\partial\theta} = Q\sum_{n=1}^{\infty}\left(\frac{R_0}{r}\right)^n \frac{P_n(0)}{n(n+2)}\sin\theta\left\{(1-\mu^2)\frac{d^2P_n}{d\mu^2} - 2\mu\frac{dP_n}{d\mu}\right\} + f'(\theta) \tag{IV-26}$$

From the second of Equation (IV-21) and from the first of Equation (IV-23) we have that

$$\frac{\partial\psi}{\partial\theta} = 2\pi r^2 u_r\sin\theta = -Q\sin\theta\sum_{n=0}^{\infty}\frac{n+1}{n+2}\left(\frac{R_0}{r}\right)^n P_n(0)P_n(\cos\theta) \tag{IV-27}$$

Comparing Equations (IV-26) and (IV-27) we get

$$Q\sum_{n=1}^{\infty}\left(\frac{R_0}{r}\right)^n \frac{P_n(0)}{n(n+2)}\sin\theta\left\{(1-\mu^2)\frac{d^2P_n}{d\mu^2} - 2\mu\frac{dP_n}{d\mu}\right\} + f'(\theta) = -Q\sin\theta\sum_{n=0}^{\infty}\frac{n+1}{n+2}\left(\frac{R_0}{r}\right)^n P_n(0)P_n(\mu)$$

Solving for $f'(\theta)$ we find

$$Q\sin\theta\sum_{n=1}^{\infty}\left(\frac{R_0}{r}\right)^n \frac{P_n(0)}{n(n+2)}\left\{(1-\mu^2)\frac{d^2P_n}{d\mu^2} - 2\mu\frac{dP_n}{d\mu} + n(n+1)P_n(\mu)\right\}$$

$$= -\frac{1}{2}Q\sin\theta - f'(\theta) \tag{IV-28}$$

Since the differential equation in the brackets is equal to zero, the above Equation (IV-28) becomes

$$f'(\theta) = -\frac{Q}{2}\sin\theta \qquad \& \qquad f(\theta) = \frac{Q}{2}(\cos\theta + C_1) \tag{IV-29}$$

and for $\theta = 0$ we have $\psi(0) = 0$. Consequently

$$C_1 = -1$$

And Equation (IV-25) for the streamfunction becomes

$$\psi(r,\theta) = \frac{Q}{2}(\cos\theta - 1) - Q\sin^2\theta \sum_{n=1}^{\infty}\left(\frac{R_0}{r}\right)^n \frac{P_n(0)}{n(n+2)}\frac{dP_n(\mu)}{d\mu} \tag{IV-30}$$

2. Case (b): $r \leq R_0$

We follow the same procedure as that for the Case (a).

From Equation (IV-20) we have that

$$u_r = \frac{\partial\phi}{\partial r} = -\frac{Q}{2\pi R_0^2}\left\{P_1(\cos\theta) + \sum_{n=0}^{\infty}{}^{*}\frac{nP_n(0)}{n-1}\left(\frac{r}{R_0}\right)^{n-1} P_n(\cos\theta)\right\}$$

$$u_\theta = \frac{\partial\phi}{\partial\theta} = \frac{Q\sin\theta}{2\pi R_0^2}\left\{\frac{dP_1}{d\mu} + \sum_{n=0}^{\infty}{}^{*}\frac{P_n(0)}{n-1}\left(\frac{r}{R_0}\right)^{n-1}\frac{dP_n(\mu)}{d\mu}\right\} \tag{IV-31}$$

Thus, the first of Equation (IV-21) becomes

$$\frac{\partial\psi}{\partial r} = -\frac{Q\sin^2\theta}{R_0}\frac{r}{R_0}\left\{\frac{dP_1}{d\mu} + \sum_{n=0}^{\infty}{}^{*}\frac{P_n(0)}{n-1}\left(\frac{r}{R_0}\right)^{n-1}\frac{dP_n(\mu)}{d\mu}\right\} \tag{IV-32}$$

Integrating we find

$$\psi = -Q\sin^2\theta\left(\frac{r}{R_0}\right)^2\left\{\frac{1}{2}\frac{dP_1}{d\mu} + \sum_{n=0}^{\infty}{}^{*}\frac{P_n(0)}{(n-1)(n+1)}\left(\frac{r}{R_0}\right)^{n-1}\frac{dP_n}{d\mu}\right\} + g(\theta) \qquad \text{(IV-33)}$$

Differentiating Equation (IV-33) with respect to θ we get

$$\frac{\partial\psi}{\partial\theta} = -Q\left(\frac{r}{R_0}\right)^2\left\{\frac{1}{2}\left(2\sin\theta\cos\theta\frac{dP_1}{d\mu} - \sin^3\theta\frac{d^2P_1}{d\mu^2}\right) + \right.$$

(IV-34)

$$\left. + \sum_{n=0}^{\infty}{}^{*}\frac{P_n(0)}{(n-1)(n+1)}\left(\frac{r}{R_0}\right)^{n-1}\left(2\sin\theta\cos\theta\frac{dP_n}{d\mu} - \sin^3\theta\frac{d^2P_n}{d\mu^2}\right)\right\} + g'(\theta)$$

From the second of Equations (IV-21) and from the first of Equations (IV-31) we have that

$$\frac{\partial\psi}{\partial\theta} = 2\pi r^2\mu_r\sin\theta$$

(IV-35)

$$= -Q\sin\theta\left(\frac{r}{R_0}\right)^2\left\{P_1(\cos\theta) + \sum_{n=0}^{\infty}{}^{*}\frac{nP_n(0)}{n-1}\left(\frac{r}{R_0}\right)^{n-1}P_n(\cos\theta)\right\}$$

Comparing Equations (IV-34) and (IV-35) we get

$$Q\left(\frac{r}{R_0}\right)^2\left\{\frac{1}{2}\sin\theta\left[2\mu\frac{dP_1}{d\mu} - (1-\mu^2)\frac{d^2P_1}{d\mu^2}\right] + \right.$$

$$\left. + \sum_{n=0}^{\infty}{}^{*}\frac{P_n(0)}{(n-1)(n+1)}\left(\frac{r}{R_0}\right)^{n-1}\sin\theta\left[2\mu\frac{dP_n}{d\mu} - (1-\mu^2)\frac{d^2P_n}{d\mu^2}\right]\right\} - g'(\theta)$$

$$= Q\sin\theta\left(\frac{r}{R_0}\right)^2\left\{P_1(\cos\theta) + \sum_{n=0}^{\infty}{}^{*}\frac{nP_n(0)}{n-1}\left(\frac{r}{R_0}\right)^{n-1}P_n(\cos\theta)\right\}$$

Solving for $g'(\theta)$ we find

$$g'(\theta) = -Q\left(\frac{r}{R_0}\right)^2 \sin\theta \left\{ \frac{1}{2}\left[(1-\mu^2)\frac{d^2P_1}{d\mu^2} - 2\mu\frac{dP_1}{d\mu} + 2P_1\right] + \right.$$

$$\left. + \sum_{n=0}^{\infty}{}^{*} \frac{P_n(0)}{(n-1)(n+1)}\left(\frac{r}{R_0}\right)^{n-1}\left[(1-\mu^2)\frac{d^2P_n}{d\mu^2} - 2\mu\frac{dP_n}{d\mu} + n(n+1)P_n\right]\right\} \tag{IV-36}$$

Since the differential equations in the brackets are equal to zero, the above Equation (IV-36) becomes

$$g'(\theta) = 0 \qquad \text{and} \qquad g(\theta) = C_2$$

and for $\theta = 0$ we have $\psi(0) = 0$. Consequently

$$g(\theta) = C_2 = 0 .$$

And Equation (IV-33) for the streamfunction becomes

$$\psi(r,\theta) = -Q\sin^2\theta\left\{\frac{1}{2}\left(\frac{r}{R_0}\right)^2 + \sum_{n=0}^{\infty}{}^{*}\frac{P_n(0)}{(n-1)(n+1)}\left(\frac{r}{R_0}\right)^{n+1}\frac{dP_n}{d\mu}\right\} \tag{IV-37}$$

because

$$\frac{dP_1}{d\mu} = \frac{d(\cos\theta)}{d(\cos\theta)} = 1$$

D. Velocity Potential at $r = R_0$

From Equations (IV-9) and (IV-20) for the velocity potential for $r \geqslant R_0$ and $r \leqslant R_0$ respectively, it is easily shown that

$$\phi(R_0, \theta)$$

has the same value on $r = R_o$.

For the proof we make use of the series expansion of the Legendre polynomial P_1 in the interval from 0 to 1 , i.e.

$$P_1 = \sum_{n=0}^{\infty} a_n P_n(\mu) \tag{IV-38}$$

Since $P_1 = \mu$ is an odd function and we are interested only in the series expansion for the interval from 0 to 1 , we define an even function $\tilde{P}_1 = |P_1|$. Thus, we have

$$\tilde{P}_1 = \sum_{n=0}^{\infty} a_n P_n(\mu)$$

Multiplying by $P_m(\mu)$ and integrating from - 1 to 1 we get

$$\int_{-1}^{+1} \tilde{P}_1 P_m(\mu) d\mu = \sum_{n=0}^{\infty} a_n \int_{-1}^{+1} P_n(\mu) P_m(\mu) d\mu$$

From the orthogonality property of the Legendre polynomials we find

$$\int_{-1}^{+1} \tilde{P} P_m(\mu) d\mu = a_m \frac{2}{2m+1} \tag{IV-39}$$

Since P_1 is positive in the interval from 0 to 1 , we may write

$$\tilde{P}_1 = |P_1| \quad \text{in } (-1 \text{ to } 0)$$
$$\tilde{P}_1 = P_1 \quad \text{in } (0 \text{ to } 1)$$

For m = even, i.e. $m = 2n$, we have

$$\int_{-1}^{+1} \tilde{P}_1 P_{2n} d\mu = 2 \int_0^1 P_1 P_{2n} d\mu = -2 \frac{P_{2n}(0)}{(2n-1)(2n+2)} \tag{IV-40}$$

Thus, from Equations (IV-39) and (IV-40) we find

$$a_{2n} = -\frac{4n+1}{(2n-1)(2n+2)} P_{2n}(0)$$

For $m =$ odd, i.e. $m = 2n+1$, we have

$$\int_{-1}^{+1} \tilde{P} P_{2n+1} d\mu = 0$$

because the integrand is an odd function. Thus, we have that

$$a_{2n+1} = 0$$

Consequently, the series expansion of P_I in the interval 0 to 1 becomes

$$P_1 = -\sum_{n=0}^{\infty} \frac{4n+1}{(2n-1)(2n+1)} P_{2n}(0) P_{2n}(\cos\theta) \qquad \text{(IV-41)}$$

And its derivative gives the following for the same interval 0 to 1 .

$$1 = -\sum_{n=0}^{\infty} \frac{4n+1}{(2n-1)(2n+1)} P_{2n}(0) \frac{dP_{2n}(\mu)}{d\mu} \qquad \text{(IV-42)}$$

Making use of Equations (IV-41) we prove that the velocity potential $\phi(r, \theta)$ given by Equations (IV-9) and (IV-20) for $r \geqslant R_o$ and $r \leqslant R_o$ respectively, has the same value on $r = R_o$.

Similarly, from Equations (IV-23) and (IV-31) for the velocities for $r \geqslant R_o$ and $r \leqslant R_o$ respectively, it is easily shown that

$$u_r(R_o, \theta) \quad \& \quad u_\theta(R_o, \theta)$$

have the same values on $r = R_o$.

For the proof we make use of Equations (IV-41) and (IV-42) respectively.

V. ANALYTICAL RESULTS

The velocity components are given by Equations (IV-23) and (IV-31) for the two regions:

$$\frac{r}{R_0} \geqslant 1 \qquad \& \qquad 0 \leqslant \frac{r}{R_0} \leqslant 1$$

For a velocity normal to the sink disk, equal to one, the strength of the source, Q , becomes

$$Q = 2\pi R_0^2 V_0 = 2\pi R_0^2$$

Thus, the above-mentioned Equations (IV-23) and (IV-31) become respectively,

a. Region $\frac{r}{R_0} \geqslant 1$

$$u_r = -\sum_{n=0}^{\infty} \frac{2n+1}{2n+2}\left(\frac{R_0}{r}\right)^{2n+2} P_{2n}(0)\, P_{2n}(\mu)$$

(V-1)

$$u_\theta = -\sin\theta \sum_{n=0}^{\infty} \frac{1}{2n+2}\left(\frac{R_0}{r}\right)^{2n+2} P_{2n}(0)\, \frac{dP_{2n}(\mu)}{d\mu}$$

b. Region $\frac{r}{R_0} \leqslant 1$

$$u_r = -\cos\theta - \sum_{n=0}^{\infty} \frac{2n}{2n-1}\left(\frac{r}{R_0}\right)^{2n-1} P_{2n}(0)\, P_{2n}(\mu)$$

(V-2)

$$u_\theta = \sin\theta \left\{ 1 + \sum_{n=0}^{\infty} \frac{1}{2n-1}\left(\frac{r}{R_0}\right)^{2n-1} P_{2n}(0)\, \frac{dP_{2n}(\mu)}{d\mu} \right\}$$

If the flow field of a propeller in a circular opening of an infinite plate is represented by a uniform sink distribution, then the flow field in the region above the plate might be represented by Equations (V-1) and (V-2).

At the sink plane, there are symmetric radial flow velocities toward the center line and a unit actual flow velocity normal to the sink disk.

In the region under the sink plane a jet with an increased total energy is emerging due to the energy added by the loading of the propeller.

For this region, one solution to the potential flow field might be found by adding a uniform flow of double the actual velocity, in the cylindrical region with cross sectional area equal to that of the sink disk.

Thus, superimposing the sink disk flow field to the uniform flow field, we get a flow field which has a jet with a contraction ratio equal to two.

Since the cylindrical flow has a discontinuity on the cylindrical surface, the resultant flow field will also have a discontinuity there.

At infinity, the mass flow from the circular opening of the plate is equal to the mass inflow from the cylindrical surface due to the sinks.

In the Tables No. V-1, V-2, and V-3, the values of the radial and tangential velocities u_r and u_θ are given for various values of the radius distance r / R_o. They have been computed from Equations (V-1) and (V-2) for the regions

$$\frac{r}{R_o} \geqslant 0 \qquad \text{and} \qquad 0 \leqslant \frac{r}{R_o} \leqslant 1$$

They give the flow field in the upper region of the plate for uniform sink distribution and normal velocity $V_o = 1$ at the sink disk. They have been programmed in the IBM 704 computer at Republic Aviation Corporation to an accuracy of ten significant figures.

In the attached Figures I and II the velocity field has been plotted as computed for uniform sink distribution. In the region under the plate, the uniform velocity equal to $2V_o = 2$ has been added geometrically, and the resultant velocity field has been plotted. In the last Figure II an effort has been made to draw the streamlines which correspond to the computed field directions.

TABLE V-1

Velocity Components u_r and u_θ

For Uniform Sink Distribution in a Circular Disk

$\cos\theta$ \ r/R_o		0	.1	.2	.3	.4	.5	.6	.7	.8	.9
.00	$-u(r)$	.000	.050	.102	.155	.213	.280	.353	.446	.572	.784
	$u(\theta)$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
.02		.020	.070	.121	.175	.233	.297	.372	.465	.590	.797
		1.000	.997	.994	.990	.986	.982	.976	.967	.952	.914
.04		.040	.090	.141	.195	.252	.316	.390	.481	.604	.796
		.999	.993	.987	.980	.972	[illegible]	[illegible]	.934	.905	.833
.06		.060	.110	.160	.214	.271	.334	.407	.496	.613	.786
		.998	.990	.980	.970	.958	.944	.927	.901	.859	.760
.08		.080	.129	.180	.232	.289	.351	.423	.509	.619	.770
		.997	.985	.972	.960	.943	.925	.903	.869	.814	.670
.10		.100	.149	.198	.250	.306	.367	.437	.519	.622	.747
		.995	.980	.964	.946	.929	.906	.877	.836	.771	.640
.20		.200	.244	.289	.335	.391	.435	.490	.549	.605	.641
		.980	.950	.920	.887	.850	.807	.755	.685	.590	.455
.30		.300	.337	.373	.410	.448	.485	.520	.550	.567	.560
		.954	.911	.866	.819	.767	.709	.640	.558	.460	.350
.40		.400	.426	.452	.476	.499	.519	.533	.537	.528	.501
		.917	.861	.805	.746	.682	.613	.537	.454	.365	.279
.50		.500	.512	.524	.533	.539	.541	.534	.519	.492	.456
		.866	.801	[illegible]	.667	.596	.522	.445	.368	.293	.225
.60		.600	.596	.590	.582	.570	.533	.529	.499	.462	.420
		.800	.728	.655	.582	.509	.435	.363	.295	.234	.181
.70		.700	.676	.651	.624	.593	.559	.520	.479	.435	.392
		.714	.639	.564	.491	.420	.352	.289	.232	.183	.143
.80		.800	.754	.707	.659	.610	.560	.509	.460	[illegible]	.368
		.600	.528	.458	.390	[illegible]	.269	.217	.173	.137	.107
.90		.900	.827	.758	.689	.621	.557	.498	.443	.393	.346
		.456	.377	.321	.268	[illegible]	.178	.142	.113	.089	.070
.925		.925	.847	.770	.695	.624	.556	.495	.438	.388	.344
		.380	.328	.277	.230	.188	.152	.121	.096	.076	.060
.950		.950	.864	.781	.707	.625	.555	.492	.434	.384	.339
		.312	.268	.226	.187	.152	.122	.097	.077	.061	.048
.975		.975	.883	.793	.707	.627	.554	.489	.430	.379	.335
		.222	.190	.159	.131	.106	.085	.067	.053	.042	.033
.990		.990	[illegible]	.799	.710	.628	.553	.487	.428	.377	.333
		.141	.120	.100	.082	.666	.053	.042	.033	.026	.021

TABLE V-2

Velocity Component u_r and u_θ

For Uniform Sink Distribution in a Circular Disk

$\cos\theta$ \ r/R_0		1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
.00	$-u(r)$	.740	.523	.406	.329	.275	.234	.202	.177	.156	.139
	$u(\theta)$	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
.02		.734	.522	.406	.329	.275	.234	.202	.177	.156	.139
		.040	.015	.008	.005	.003	.002	.002	.001	.001	.001
.04		.719	.519	.404	.328	.274	.233	.202	.176	.156	.139
		.076	.029	.015	.009	.006	.004	.003	.002	.002	.001
.06		.698	.513	.402	.327	.273	.233	.201	.176	.156	.139
		.105	.043	.023	.013	.009	.006	.005	.003	.003	.002
.08		.673	.505	.398	.325	.272	.232	.201	.176	.155	.139
		.127	.055	.030	.018	.012	.008	.006	.004	.003	.003
.10		.647	.496	.394	.323	.271	.231	.200	.176	.155	.138
		.144	.066	.036	.022	.015	.010	.007	.006	.004	.003
.20		.536	.444	.367	.308	.261	.225	.196	.172	.153	.137
		.170	.098	.060	.039	.026	.019	.014	.010	.008	.006
.30		.460	.395	.337	.288	.248	.216	.189	.167	.149	.134
		.160	.105	.070	.048	.034	.025	.018	.014	.011	.009
.40		.407	.353	.309	.268	.234	.206	.182	.162	.145	.130
		.143	.100	.071	.051	.037	.028	.021	.017	.013	.010
.50		.367	.324	.284	.250	.220	.195	.174	.155	.140	.126
		.123	.090	.066	.050	.037	.029	.022	.018	.014	.011
.60		.336	.298	.264	.234	.207	.185	.166	.149	.134	.122
		.104	.078	.060	.046	.035	.028	.022	.017	.014	.011
.70		.312	.277	.246	.219	.196	.176	.158	.143	.129	.118
		.085	.065	.051	.040	.031	.025	.020	.016	.013	.011
.80		.292	.259	.231	.207	.185	.167	.151	.137	.124	.113
		.065	.051	.040	.032	.026	.021	.017	.014	.011	.009
.90		.274	.245	.219	.196	.176	.159	.144	.131	.120	.109
		.043	.034	.028	.022	.018	.015	.012	.010	.008	.007
.925		.271	.241	.216	.193	.174	.157	.143	.130	.118	.108
		.037	.030	.024	.019	.016	.013	.011	.009	.007	.006
.950		.267	.238	.213	.191	.172	.155	.141	.128	.117	.107
		.030	.024	.019	.016	.013	.010	.009	.007	.006	.005
.975		.263	.235	.210	.189	.170	.154	.140	.127	.116	.106
		.021	.017	.014	.011	.009	.007	.006	.005	.004	.004
.990		.261	.233	.208	.187	.169	.153	.139	.126	.116	.106
		.013	.011	.009	.007	.006	.005	.004	.003	.003	.002

TABLE V-3

Velocity Components u_r and u_θ

For Uniform Sink Distribution in a Circular Disk

$\cos\theta$ \ r/R_0		.92	.94	.95	.96	.98	1.02	1.04	1.05	1.06	1.08
.00	$-u(r)$ $u(\theta)$	.852 1.000	.940 1.000	.997 1.000	1.004 1.000	1.278 1.000		1.035 .000	.963 .000	.904 .000	.911 .000
.02		.662 .897	.942 .870		1.048 .821			1.003 .117		.889 .015	.838 .053
.04		.852 .803	.915 .758		.995 .689			.934 .190		[illegible] .134	.780 .099
.06		.829 .722	.875 .670		.917 .602			.864 [illegible]		.865 .172	.749 .133
.08		.801 .654	.832 .603		.955 .539			.804 .247		.760 .195	.716 .157
.10		.777 .595	792 .549	.800 .522	[illegible] [illegible]			.754 254	.737 .230	.720 .209	.683 .172
.20				.639 .31[illegible]					.532 .226		
.30				.544 .295					.492 [illegible]		
.40				.491 .239					.453 .170		
.50				.425 .195					.390 .144		
.60				.399 .[illegible]58					.357 .120		
.70				.371 .125					.331 .097		
.80				.348 .093					.319 .074		
.90				.325 .062							

VI. APPENDIX I

Wing With Line Sink on the Upper Surface

In this article a general method will be given for the evaluation of the velocity field of a flat wing with a line sink on the upper surface. The chord of the wing is taken to be of length $c = 4$.

First we evaluate the complex velocity potential.

We assume that the wing is in the z-plane and has a forward velocity W under an angle of attack α_0.

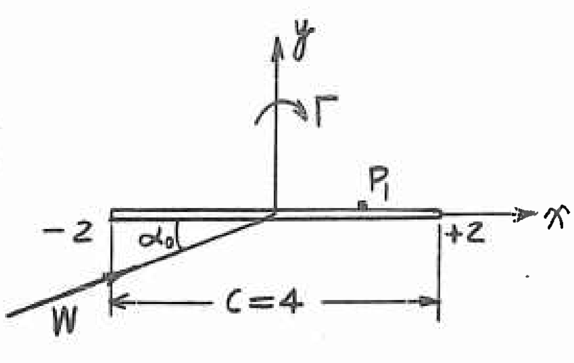
We make the transformation

$$z = z' + \frac{1}{z'} \tag{VI-1}$$

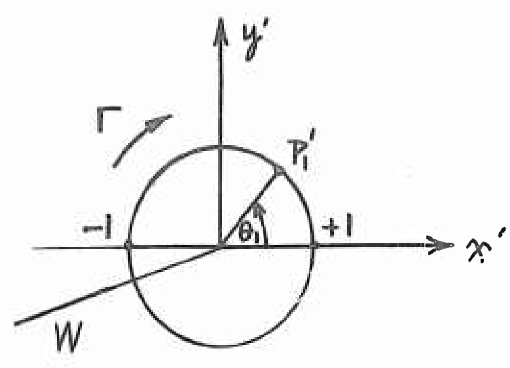
where:

$$z = x + iy \quad \& \quad z' = x' + iy'$$

Which makes the wing a cylinder of unit radius.



z-plane



z'-plane

Every point on the flat wing is mapped into a point on the unit circle.

A sink at the point $P_1\ (x_1\ ,\ 0_+)$ on the flat wing in the z-plane, is mapped as a sink at the point $P_1'\ (x_1'\ ,\ y_1')$ on the unit circle in the z'-plane.

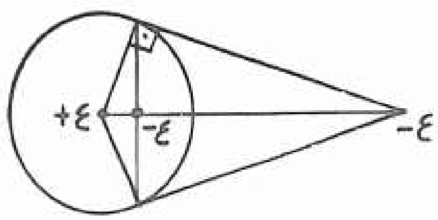
where:

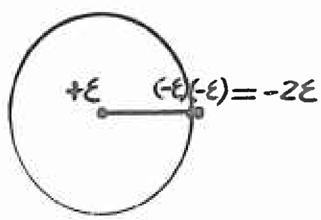
$$z_1' = x_1' + i\sqrt{1 - x_1'^2} \qquad z_1 = z_1' + \frac{1}{z_1'} = e^{i\theta_1} + e^{-i\theta_1} = 2\cos\theta_1$$

or

$$x_1 = 2\cos\theta_1 \qquad \& \qquad y_1 = 0_+ \tag{VI-2}$$

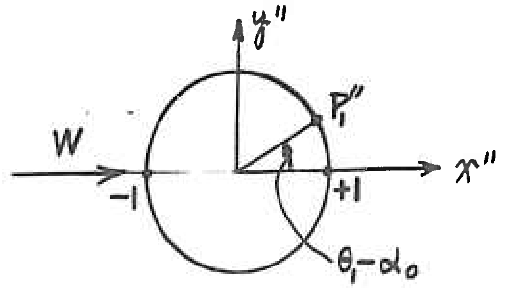
From the theory of the flow around a cylinder due to a sink of given strength, we have that the boundary condition of zero velocity across the solid cylindrical surface is satisfied, by replacing the cylinder with another sink and a source of the same strength as the given sink, placed respectively at the inverse point with respect to the circle and at the axis of the cylinder.





Thus, the flow due to a sink or a source on the cylindrical surface would satisfy the boundary condition if we add a source or a sink of half strength respectively, at the axis.

Rotating the z'-plane by an angle equal to $-\alpha_0$ (where α_0 is the angle of attack) we get:



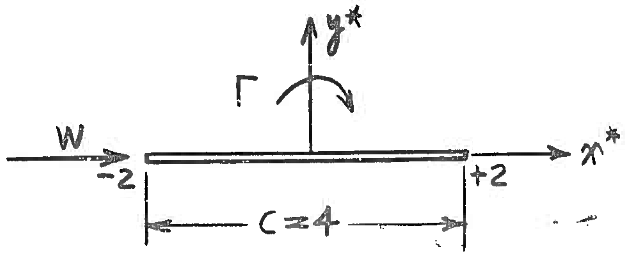
$$z''=z'e^{-i\alpha_0} \tag{VI-3}$$

z''-plane

and

$$x_1''=\cos(\theta_1-\alpha_0)=\cos(\cos^{-1}x_1'-\alpha_0) \tag{VI-4}$$

Finally the transformation $W=z''+\dfrac{1}{z''}$ maps the z''-plane into a w-plane with the cylinder transformed into a flat plate, with the sink at the point P_1^* $(x_1^*, 0_+)$.



$$W=z''+\frac{1}{z''} \tag{VI-5}$$

w-plane

where

$$x_1^*=2\cos(\theta_1-\alpha_0)=2\cos(\cos^{-1}x_1'-\alpha_0) \tag{VI-6}$$

or

$$x_1^*=2x_1''=2\left(x_1'\cos\alpha_0+\sqrt{1-x_1'^2}\,\sin\alpha_0\right)$$

The complex velocity potential for a flow around the unit cylinder with a sink of power 2ε at the point z_1'' and a circulation Γ, is given by

$$F(z'')=W\left(z''+\frac{1}{z''}\right)-\frac{2\varepsilon}{2\pi}\ln(z''-z_1'')+\frac{\varepsilon}{2\pi}\ln z''+\frac{i\Gamma}{2\pi}\ln z'' \qquad \text{(VI-7)}$$

where the third term corresponds to the potential of a source at the axis of the cylinder with half the strength of the given sink, for the satisfaction of the boundary conditions.

Making use of Equation (VI-3), the above equation reads as follows in the z''-plane:

$$F(z')=W\left(z'e^{-i\alpha_0}+\frac{e^{i\alpha_0}}{z'}\right)-\frac{\varepsilon}{\pi}\ln(z'-z_1')+\frac{\varepsilon}{2\pi}\ln z'+\frac{i\Gamma}{2\pi}\ln z'+\text{Const.} \qquad \text{(VI-8)}$$

The Kutta condition gives that the velocity at the trailing edge should be finite and we have that

$$\frac{dF}{dz}=\frac{dF}{dz'}\frac{dz'}{dz} = \text{finite at } z'=1$$

Since at $z'=1$

$$\frac{dz'}{dz}=\frac{1}{\frac{dz}{dz'}}=\frac{1}{1-\frac{1}{z'^2}}\longrightarrow\frac{1}{0}\longrightarrow\infty$$

we should have

$$\frac{dF}{dz'}=0 \qquad \text{at } z'=1$$

This gives the value of the circulation strength Γ .

Thus, differentiating Equation (VI-8) , we get:

$$\frac{dF}{dz'} = W\left(e^{-i\alpha_0} - \frac{e^{i\alpha_0}}{z'}\right) - \frac{\varepsilon}{\pi}\frac{1}{z'-z_1'} + \frac{\varepsilon}{2\pi}\frac{1}{z'} + \frac{i\Gamma}{2\pi z'} \tag{VI-10}$$

and at $z' = 1$ the above equation becomes

$$\left.\frac{dF}{dz'}\right|_{z'=1} = -2iW\sin\alpha_0 - \frac{\varepsilon}{\pi}\frac{1}{1-z_1'} + \frac{\varepsilon}{2\pi} + \frac{i\Gamma}{2\pi} = 0 \tag{VI-11}$$

where

$$z_1' = x_1' + i\sqrt{1 - x_1'^2}$$

Solving the above equation for Γ we get:

$$\Gamma = 4\pi W\sin\alpha_0 + \varepsilon\sqrt{\frac{1+x_1'}{1-x_1'}} \tag{VI-12}$$

and substituting x_1' by $\cos\ \theta_1$ the above equation becomes

$$\Gamma = 4\pi W\sin\alpha_0 + \varepsilon\frac{\sin\theta_1}{1-\cos\theta_1} \tag{VI-13}$$

From the equation for the complex potential

$$\frac{dF}{dz} = u - iv \tag{VI-14}$$

and from Equations (VI-9) and (VI-10) we get:

$$u - iv = \left\{ W\left(e^{-i\alpha_0} - \frac{e^{i\alpha_0}}{z'^2}\right) - \frac{\varepsilon}{\pi}\frac{1}{z'-z'_1} + \frac{\varepsilon}{2\pi z'} + \frac{i\Gamma}{2\pi z'} \right\} \frac{1}{1-\frac{1}{z'^2}} \tag{VI-15}$$

On the wing the complex velocity becomes

$$u - iv = \left\{ W\left(e^{-i\alpha_0} - \frac{e^{i\alpha_0}}{e^{2i\theta}}\right) - \frac{\varepsilon}{\pi}\frac{1}{e^{i\theta}-e^{i\theta_1}} + \frac{\varepsilon}{2\pi e^{i\theta}} + \frac{i\Gamma}{2\pi e^{i\theta}} \right\} \frac{1}{1-e^{2i\theta}} \tag{VI-16}$$

where θ_1 is given by $x_1 = 2\cos\theta_1$

Simplifying the above equation and making use of the identity

$$\frac{1}{1-e^{i(\theta_1-\theta)}} = \frac{1}{2}\left[1 + i\,\frac{\sin(\theta_1-\theta)}{1-\cos(\theta_1-\theta)}\right] \tag{VI-17}$$

The above Equation (VI-16) becomes

$$u - iv = \left[W\sin(\theta-\alpha_0) - \frac{\varepsilon}{4\pi}\frac{\sin(\theta_1-\theta)}{1-\cos(\theta_1-\theta)} + \frac{\Gamma}{4\pi} \right] \frac{1}{\sin\theta} \tag{VI-18}$$

Substituting for Γ from Equation (VI-13) we get

$$u = \frac{1}{\sin\theta}\left[W\left\{\sin(\theta-\alpha_0) + \sin\alpha_0\right\} + \frac{\varepsilon}{4\pi}\left\{\frac{\sin\theta_1}{1-\cos\theta_1} - \frac{\sin(\theta_1-\theta)}{1-\cos(\theta_1-\theta)}\right\} \right] \tag{VI-19}$$

$$v = 0$$

where the actual strength of the sink is $\mathcal{E}$

$$\mathcal{E} = \frac{1}{2}(2\mathcal{E})$$

because only the strength of the sink corresponding to the flow field outside the cylindrical surface is concerned.

From the trigonometric identity

$$\frac{\sin(\theta_1 - \theta)}{1 - \cos(\theta_1 - \theta)} = \frac{\sin\theta_1 + \sin\theta}{\cos\theta - \cos\theta_1} \qquad \text{(VI-20)}$$

the above Equation (VI-19) is written as

$$u = \frac{1}{\sin\theta}\left\{W\left[\sin(\theta - \alpha_0) + \sin\alpha_0\right] + \frac{\mathcal{E}}{4\pi}\left[\frac{\sin\theta_1}{1 - \cos\theta_1} - \frac{\sin\theta_1 + \sin\theta}{\cos\theta - \cos\theta_1}\right]\right\} \qquad \text{(VI-21)}$$

For the case of very small angles of attack α_0, we might write

$$\sin(\theta - \alpha) \simeq \sin\theta - \alpha_0 \cos\theta \qquad \text{and} \qquad \sin\alpha \simeq \alpha$$

and the above Equation (VI-21) reads as follows:

$$u = \frac{1}{\sin\theta}\left\{W\left[\alpha_0(1 - \cos\theta) + \sin\theta\right] + \frac{\mathcal{E}}{4\pi}\left[\frac{\sin\theta_1}{1 - \cos\theta_1} - \frac{\sin\theta_1 + \sin\theta}{\cos\theta - \cos\theta_1}\right]\right\} \qquad \text{(VI-22)}$$

This equation for the velocity can be separated into even and odd terms as follows:

$$u = (V + V_s) + (V_\Gamma + V_n)$$

Even:

$$\begin{cases} V = W \\ V_s = \dfrac{\varepsilon}{4\pi} \dfrac{1}{\cos\theta_1 - \cos\theta} & \text{(symmetric sinks)} \end{cases}$$

Odd:

$$\begin{cases} V_\Gamma = \left[\dfrac{\varepsilon}{4\pi} \dfrac{\sin\theta_1}{1-\cos\theta_1} + W\alpha_0(1-\cos\theta)\right]\dfrac{1}{\sin\theta} & \text{(circulation)} \\ V_n = \dfrac{\varepsilon}{4\pi}\left(\dfrac{\sin\theta_1}{\cos\theta_1 - \cos\theta}\right)\dfrac{1}{\sin\theta} & \text{(anti-symmetric sink and source)} \end{cases}$$

(VI-23)

In the case of zero angle of attack, i.e., $\alpha_0 = 0$, the above Equations (VI-23) become

$$V = W = \frac{\varepsilon}{2\pi} W_0$$

$$V_s = \frac{\varepsilon}{4\pi} \frac{1}{\frac{x_1}{2} - \frac{x}{2}} = \frac{\varepsilon}{2\pi} \frac{1}{x_1 - x} \qquad \text{(VI-24)}$$

$$V_\Gamma = \frac{\varepsilon}{4\pi} \frac{\frac{\sqrt{2^2 - x_1^2}}{2 - x_1}}{\frac{1}{2}\sqrt{2^2 - x^2}} = \frac{\varepsilon}{2\pi} \frac{1}{\sqrt{2^2 - x^2}} \sqrt{\frac{2 + x_1}{2 - x_1}}$$

$$V_n = \frac{\varepsilon}{4\pi} \frac{\frac{1}{2}\sqrt{2^2 - x_1^2}}{\frac{x_1}{2} - \frac{x}{2}} \frac{1}{\frac{1}{2}\sqrt{2^2 - x^2}} = \frac{\varepsilon}{2\pi} \frac{1}{x_1 - x} \sqrt{\frac{2^2 - x_1^2}{2^2 - x^2}}$$

where we made use of the equations

$$\cos\theta_1 = \frac{x_1}{2} \,, \quad \cos\theta = \frac{x}{2} \,, \quad \sin\theta = \frac{1}{2}\sqrt{2^2 - x^2} \;\;\&\;\; \sin\theta_1 = \frac{1}{2}\sqrt{2^2 - x_1^2}$$

The above Equations (VI-24) are identical with Equations (III-9a) of Chapter III. The only difference is that in Equations (III-9a) the chord of the wing has length $c = 2$, and in Equations (VI-24) the chord is $c = 4$.

VII. APPENDIX II

POWER FOR SUSTAINED LEVEL FLIGHT OF VTO AIRCRAFT

It has been observed that in most cases so far experimentally investigated that the power consumed in transition is actually greater than that in hovering flight. This has been due to various, more-or-less unexpected difficulties. The undesirable pitching moment is cancelled by a negative lift on the aircraft causing an increase in the body drag, with an induced drag in the bargain. The importance of a wing surface as a relieving device has been pointed out previously, also the absolute necessity of employing proper baffles to avoid serious losses in propeller efficiency and avoid destruction of the propeller.

We will, in the following, as an example only, study a case in which the outlet is designed with variable deflectors and the area perpendicular to the thrust vector is reduced as the cosine of the thrust vector with the vertical. This arrangement represents the simplest mechanical case. For instance, when the velocity is pointing rearwards at an angle of 60 degrees with the vertical the free cross section of the openings is one half of the original full area. In such a case the pressure behind the propeller and ahead of the outlet baffles will increase and the flow velocity through the propeller decrease with a corresponding reduction in the mass flow. An adjustable pitch propeller is required and only a limited range of operation is available. We shall, only briefly, indicate a second case in which the mass flow is kept constant thus permitting the propeller to operate at constant pitch under full power for acceleration while the outlet openings are controlled to maintain a constant mass flow. In this case, the free area perpendicular to the reaction vector is slightly reduced as the speed is increased. Details of this case will not be considered in the present paper. However, it is pointed out that a complete survey of all possible methods is desirable and should be undertaken in the future.

Finally, there is a short discussion of the pitching moment and required compensation.

It will be shown in the following how the main performance parameters are related in the case for which the outlet area perpendicular to the direction of the jet is varied as the cosinus of the rearward deflection angle.

Let us express the lifting force of the fan or fans by the formula

$$T = M_S V_D \tag{VII-1}$$

where V_D is the vertical downward velocity at the outlet of the duct. M_s is the mass of air per second and is thus equal to

$$M_S = \rho A_D V_D \tag{VII-2}$$

where ρ is the air density and A_D is the cross section of the duct or ducts at the outlet, measured horizontally or at right angles to the vertical outlet velocity V_D. Thus one has finally also

$$T = \rho A_D V_D^2 \tag{VII-3}$$

Let us consider the condition in a forward equilibrium flight at a velocity V. The expression for T remains as given. The aircraft has a wing of area S. The drag coefficient of the aircraft is then expressed as

$$C_D = C_{DO} + \frac{C_L^2}{\pi A} \tag{VII-4}$$

where, as usual, C_{DO} is the drag at zero lift and C_L the lift coefficient of the wing. The last term gives the induced drag of the wing surface S of aspect ratio A. The drag force on the aircraft in forward flight is thus

$$D = \frac{1}{2} \rho V^2 S C_D \tag{VII-5}$$

and the lift force is

$$L = \frac{1}{2} \rho V^2 S C_L \qquad \text{(VII-6)}$$

where then this lift force of the wing system plus the thrust force T of the fan system must be equal to the weight W of the aircraft or

$$L + T = W \qquad \text{(VII-7)}$$

It can be verified immediately that the lift of the wing, if any, is far less costly in terms of power than the lift due to the fans. Therefore, as will be evident in the following, the lift coefficient C_L should, under all circumstances, be maintained constant at the highest possible safe or convenient value until T, the lift of the fan, goes to zero.

Let us consider the ideal power-consumption of an aircraft in uniform flight with a drag coefficient C_D and a lift coefficient C_L. The theoretical work done on the mass M_s in unit time is

$$W_i = \frac{1}{2} M_S (V_o^2 - V^2)$$

where V_o ~~is~~ the outlet velocity which we may consider as composed of the downward component V_D already defined and a rearward pointing component V_R. We may, therefore, rewrite the equation as

$$W_i = \frac{1}{2} M_S (V_D^2 + V_R^2 - V^2) \qquad \text{(VII-8)}$$

Notice, at this point, that in addition to the momentum drag on the aircraft there is, of course, a drag due to C_D. There is a useful thrust produced by the fan system equal to

$$T = M_s (V_R - V)$$

and the useful work required would be equal to

$$W_f = M_s (V_R - V) V \qquad \text{(VII-9)}$$

By subtracting the useful work given by Equation (VII-9) from the total theoretical work of the propeller given by Equation (VII-8) and differentiating with respect to V_R, we find the condition for the minimum work required,

$$V_R = V \qquad \text{(VII-10)}$$

and we have the following important statement. The minimum work is expendent, if and when the rearward velocity component of the issuing jet is identical with the flight velocity. The drag caused by the normal resistance of the airplane is a function of V only and does not affect the derivation.

The result is perfectly general and to be expected since it obviously means that the horizontal momentum drag of the air mass is avoided when $V_R = V$.

Note also that the wing should, under all conditions, be maintained in a position of the highest permissible lift coefficient. We shall next consider this problem more fully, remembering that thrust must also be provided to take care of the normal drag of the aircraft. We have already shown that the momentum drag of the aircraft is "avoided" when, and if, $V_R = V$. It is, however, necessary to provide a net or real forward thrust force produced by the fan system in the magnitude (Equation (VII-5) and a net lift force of the fan system equal to the excess of the weight W of the aircraft over the lift of the wing (Equation (VII-6).

Therefore, with a thrust

$$T_R = M_S\,(V_R - V) \tag{VII-11}$$

one has the equations

$$T_R = D$$ the drag of the aircraft

and $$T = W - L$$ or written out

for the drag

$$\rho V_D A_D (V_R - V) = \frac{1}{2}\rho C_D V^2 S \tag{VII-12}$$

and for the lift

$$\rho V_D^2 A_D + \frac{1}{2}\rho C_L V^2 S = W \tag{VII-13}$$

With W, C_L, C_D, S, and A_D given in these two basic equations one has from the first equation

$$\frac{V_R}{V_D} = \frac{1}{2} C_D \frac{S}{A_D}\left(\frac{V}{V_D}\right)^2 + \frac{V}{V_D} \tag{VII-14}$$

where the velocities are given in terms of the velocity V_D.

The second equation (VII-13) is similarly written in non-dimensional form:

$$\frac{1}{2} C_L \frac{S}{A_D}\left(\frac{V}{V_D}\right)^2 = \frac{W}{\rho V_D^2 A_D} - 1 \tag{VII-15}$$

Here again the velocities are given in terms of the exit velocity V_D.

Work done by the power plant in equilibrium forward flight is again given by the formula (Equation (VII-8)) which may be written in the form

$$W_i = \frac{1}{2}\rho V_D A_D (V_D^2 + V_R^2 - V^2)$$

where V_R given by the expression (Equation (VII-14)) balances the drag of the aircraft. Any excess power is or may be used for acceleration of the aircraft.

Rewritten in non-dimensional form and rearranged

$$\left(\frac{V_R}{V_D}\right)^2 - \left(\frac{V}{V_D}\right)^2 = \frac{W_i}{\frac{1}{2}\rho A_D V_D^3} - 1 \tag{VII-16}$$

Notice in these formulas that in stationary condition V_R and V are both equal to zero and both sides of Equations (VII-15) and (VII-16) are equal to zero

Equations (VII-15) and (VII-16) may be rewritten

$$\alpha \frac{C_L}{C_D}\left(\frac{V}{V_D}\right)^2 = \frac{W}{T} - 1 \tag{VII-17}$$

and

$$\left(\frac{V_R}{V_D}\right)^2 - \left(\frac{V}{V_D}\right)^2 = \frac{W_i}{\frac{1}{2} T V_D} - 1 \tag{VII-18}$$

where $\alpha = \frac{1}{2} C_D \frac{S}{A_D}$, T is the vertical thrust and it will, of course, decrease in forward motion.

Inserting Equation (VII-14) in Equation (VII-18) there results

$$\left[\alpha\left(\frac{V}{V_D}\right)^2 + \frac{V}{V_D}\right]^2 - \left(\frac{V}{V_D}\right)^2 = \chi - 1 \tag{VII-19}$$

where

$$\chi = \frac{W_i}{\frac{1}{2} T V_D}$$

We have now in final form three equations; the drag equation (VII-12) or (VII-14)

$$\frac{V_R}{V_D} = \alpha\left(\frac{V}{V_D}\right)^2 + \frac{V}{V_D} \tag{VII-19a}$$

the lift equation (VII-13) or (VII-15)

$$\alpha \frac{C_L}{C_D}\left(\frac{V}{V_D}\right)^2 = \frac{W}{T} - 1 = \lambda$$

and finally, the work relation (VII-16) or (VII-18)

$$\left(\frac{V_R}{V_D}\right)^2 - \left(\frac{V}{V_D}\right)^2 = \frac{W_i}{\frac{1}{2} T V_D} - 1$$

By inserting the first equation into the third one obtains Equation (VII-19) above. The quantity χ may then be plotted against $\frac{V}{V_D}$ to give a universal plot. Next, the second equation gives a similar plot of λ against the same abscissa $(\frac{V}{V_D})$. Actually the second equation gives us the value of T and consequently the value of V_D for any point $(\frac{V}{V_D})$. The second curve gives finally, with T and V_D known, the power consumption W_i for each point $(\frac{V}{V_D})$. With all the constants given one may finally plot W, T, and V_D against the forward velocity V. Excess power is then used for acceleration. The ideal excess power is given as a function of the forward velocity. Since the value of C_L has been kept constant in this treatment, the above equations apply to the region below the velocity for which the aircraft is completely supported by a wing.

VIII. APPENDIX III

NOSE-UP PITCHING MOMENT

It is assumed that the airplane in forward flight with the fan operating maintains a lift coefficient C_L of a constant value. Since part of the lift is indirectly caused by an increase in the circulation due to the effect of the jet this result means that the actual angle of attack will be gradually adjusted, preferably by automatic means providing action on ailerons and tail surface to maintain, at all times, the largest possible lift on the wing proper -- at least to a practicable extent. It is useful to develop here the fraction of thrust carried by the fan system as a fraction of the total weight W. From Equation (VII-15) it may be seen that

$$\frac{W}{T} - 1 = \beta\left(\frac{V}{V_D}\right)^2$$

where $\beta = \frac{1}{2} C_L \frac{S}{A_D}$ and the fraction is

$$\gamma = \frac{T}{W} = \frac{1}{1+\beta\left(\frac{V}{V_D}\right)^2} \qquad \text{(VIII-1)}$$

The ratio $\gamma = \frac{T}{W}$ is thus a function of the forward velocity V for a given aircraft with a given fixed horsepower.

The pitching moment caused by the incoming air on the surface surrounding the inlet duct is shown in the theoretical section (IIB) to be expressed as

$$M_2 = T \frac{V}{V_D} \frac{1}{2} R \qquad \text{(VIII-2)}$$

where R in simple terms is the "radius" of the related wing area around the duct. In other words, the effective arm of the incoming mass force acts as if located $\frac{1}{2}$ R above the duct inlet.

There is, however, also a nose-up pitching moment due to the rearward velocity V_R of the issuing jet, if and when, the jet deflectors are located below the center of drag of the aircraft which is normally the case. The effective total moment of incoming and outgoing mass forces is thus

$$M = T \frac{V}{V_D} \left(\frac{1}{2} R + a \right) \qquad \text{(VIII-3)}$$

where the arm a is the distance of the location of the jet deflectors below the duct entrance (or the wing surface).

The remainder of the forward thrust force

$$T \frac{V_R - V}{V_D}$$

acts to counterbalance the drag of the aircraft. The arm of this force is, therefore, equal to the distance of the jet-deflectors below the center of resistance of the aircraft. This center may or may not be near the center of gravity.

Let this arm be b and we have a nose-up pitching moment

$$M = T\,\frac{V_R - V}{V_D}\,b \qquad \text{(VIII-4)}$$

The counterbalancing moment, for instance, from a tail surface S_T with an effective arm L and a lift coefficient C_{LT} is

$$M_e = \frac{1}{2}\rho\, C_{LT} V^2 S_T L$$

which may be written

$$M_e = \frac{1}{2}\rho V_D^2 A_D \left(\frac{V}{V_D}\right)^2 \frac{S_T}{A_D} L\, C_{LT}$$

or

$$M_e = T\left(\frac{V}{V_D}\right)^2 \varepsilon L \qquad \text{(VIII-5)}$$

where

$$\varepsilon = \frac{1}{2} C_{LT} \frac{S_T}{A_D}$$

With Equations (VIII-3), (VIII-4), and (VIII-5) there is

$$\frac{V}{V_D}\left(\frac{1}{2}R + a\right) + \frac{V_R - V}{V_D}\,b = \left(\frac{V}{V_D}\right)^2 \varepsilon L$$

inserting the expression for $\dfrac{V_R - V}{V_D}$ from Equation (VII-19a) we get

$$\frac{V}{V_D}\left(\frac{1}{2}R + a\right) + \alpha\left(\frac{V}{V_D}\right)^2 b = \varepsilon\left(\frac{V}{V_D}\right)^2 L$$

or

$$\varepsilon = \frac{1}{L}\,\frac{\alpha\left(\frac{V}{V_D}\right)^2 b + \left(\frac{1}{2}R + a\right)\frac{V}{V_D}}{\left(\frac{V}{V_D}\right)^2} \qquad \text{(VIII-6)}$$

Notice at once that the condition $V = 0$ calls for, as expected, an infinite tail effect. For very large forward speed $V >> V_D$

$$\varepsilon = \alpha \frac{b}{L}$$

Since the arm b between the center of drag and the location of the deflection vanes may be quite small, it is seen that a rather normal or small tail surface will suffice. The quantity V_D is, of course, at high speed in itself small (while V_R or the rearward velocity is large).

The thrust carried by the propeller as lift is, in this case, negligible and the aircraft is flying as a conventional airplane.

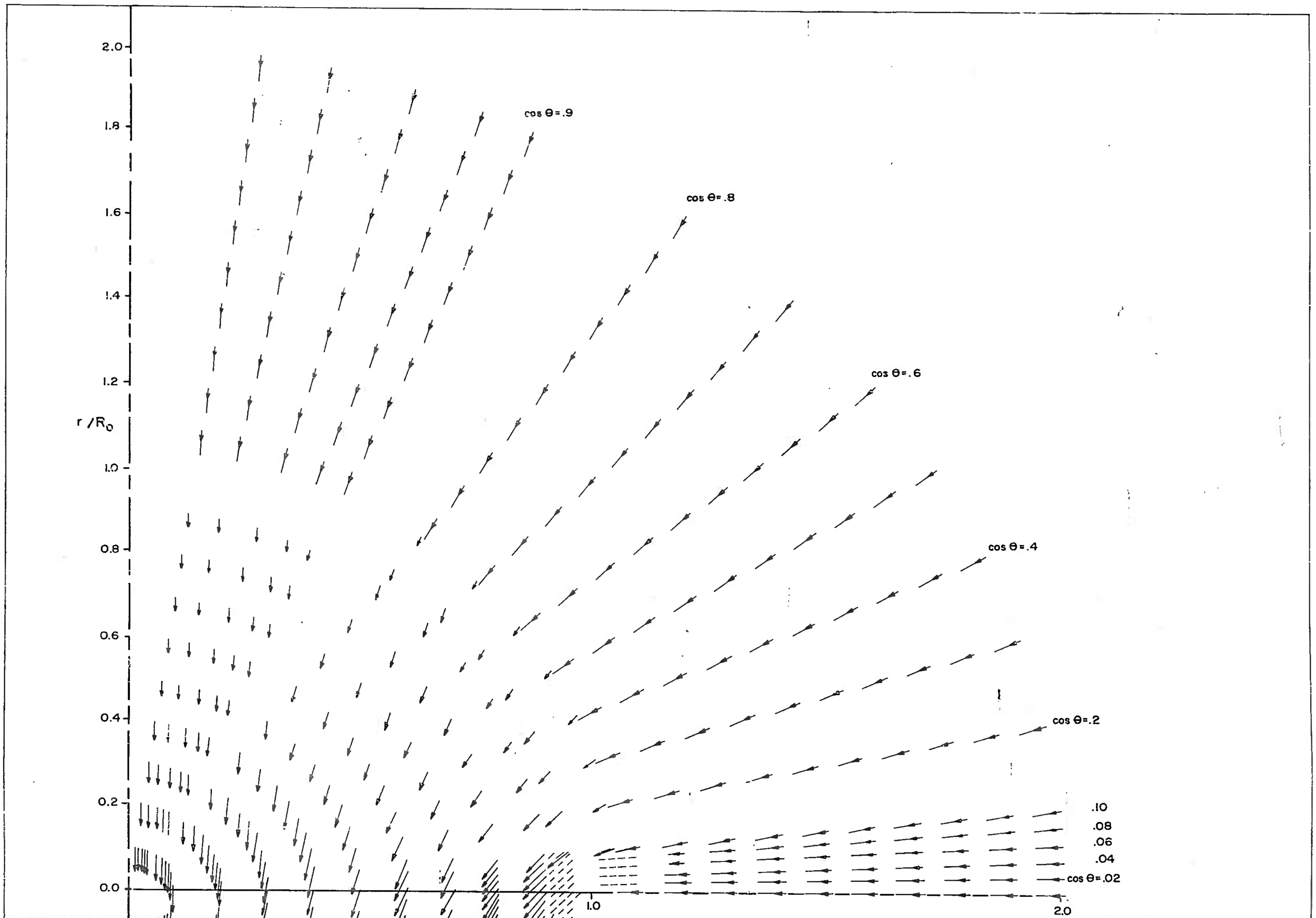
Further, if α is in the order of five or so, there is no serious requirement on the tail for $V = V_D$. For small forward speed, however, we may not use a tail surface for compensation of the pitch-up moment since then, of course, the tail is completely ineffective.

There remains, however, the possibility to use a small tail rotor à la the helicopter except that it is to be designed for vertical operation. It will, in fact, be shown that the power required for such a scheme is actually quite negligible and that the scheme is practical. We need only balance the moment (Equation (VIII-5)).

$$M = T \frac{V}{V_D} \left(\frac{1}{2} R + a \right)$$

Since the moment this time, fortunately, is a pitch-up moment, the location of the tail rotor would be in the horizontal tail surface and the thrust would be downward so that any thrust carried would also unload the main rotor by the same amount.

Notice that in the proposed aircraft the requirements outlined are achieved by various arrangements of simple nature. The aircraft is designed as a high speed machine. It has a normal tail fully adequate for flight conditions and large enough to balance required rearward thrust of the jets at a nominal forward speed. It has a small tail rotor or supply duct to the tail using in the order of one percent of the total power of the aircraft. This element fully takes care of the pitch-up moment inherent in this type of aircraft at a negligible loss of efficiency compared to any other scheme for the same purpose. Such element is further highly desirable and necessary for zero speed control in pitch and the aircraft control system should be supplemented by similar duct or ducts to one or both wing tips to provide lateral control at zero and low speed. The outlets of said ducts to be provided with adjustable-area nozzles to achieve highest efficiency. Power or air to be supplied to such rotors or ducts either from the main powerplant or duct or from a separately powered compressor or by mechanical transmission to small fans as may be preferred.



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.04
cos θ = .02

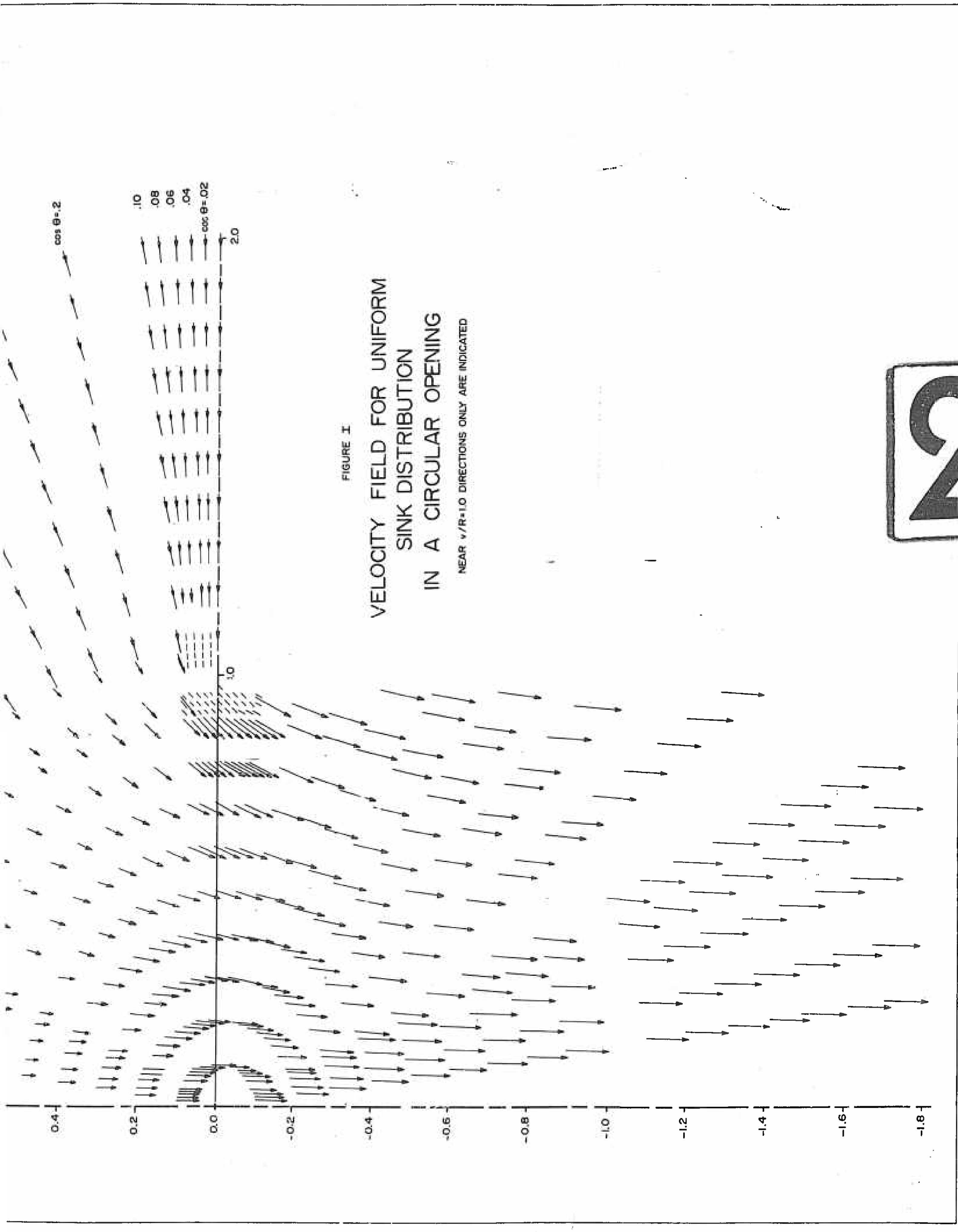
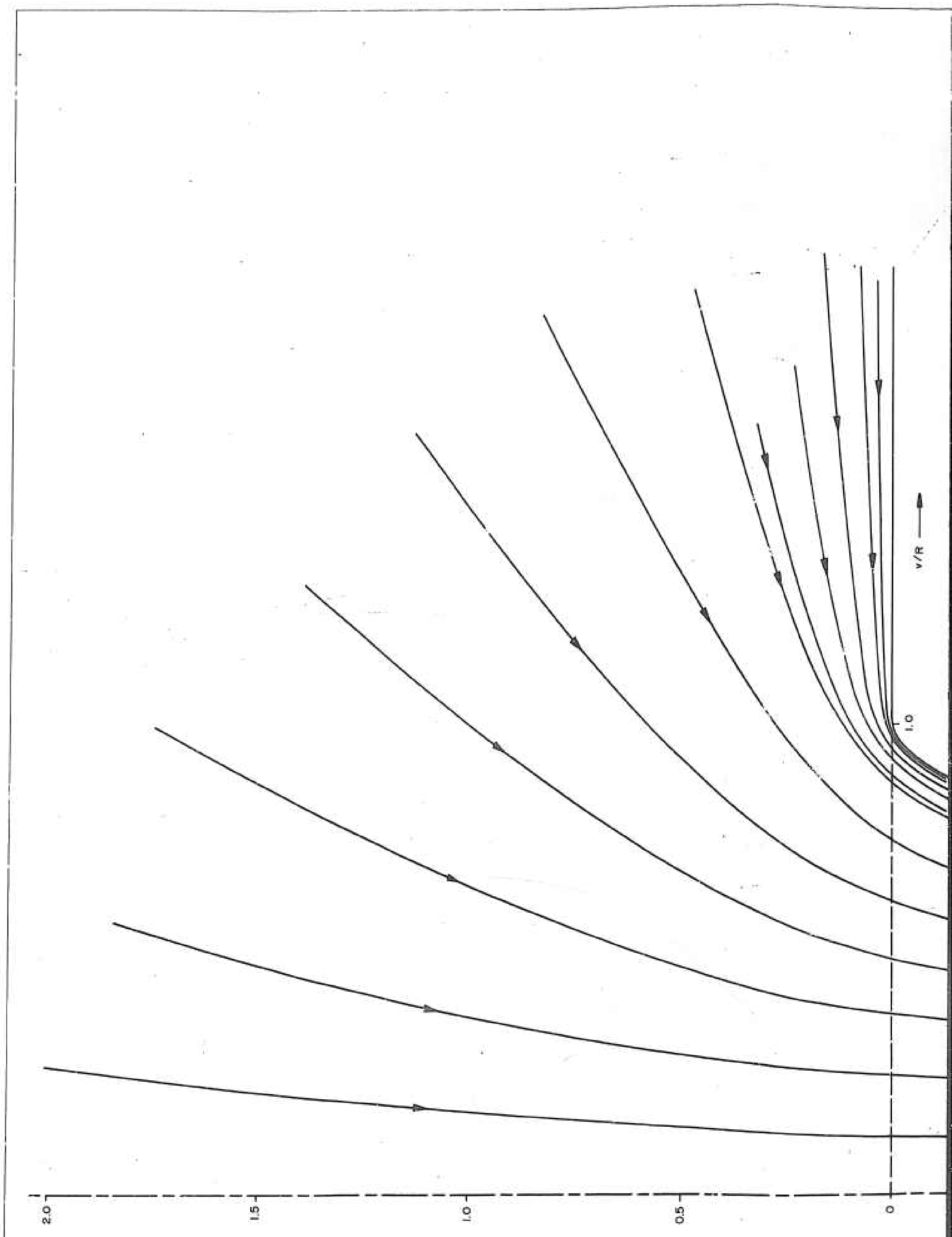


FIGURE I

VELOCITY FIELD FOR UNIFORM SINK DISTRIBUTION IN A CIRCULAR OPENING

NEAR v/R=1.0 DIRECTIONS ONLY ARE INDICATED



1

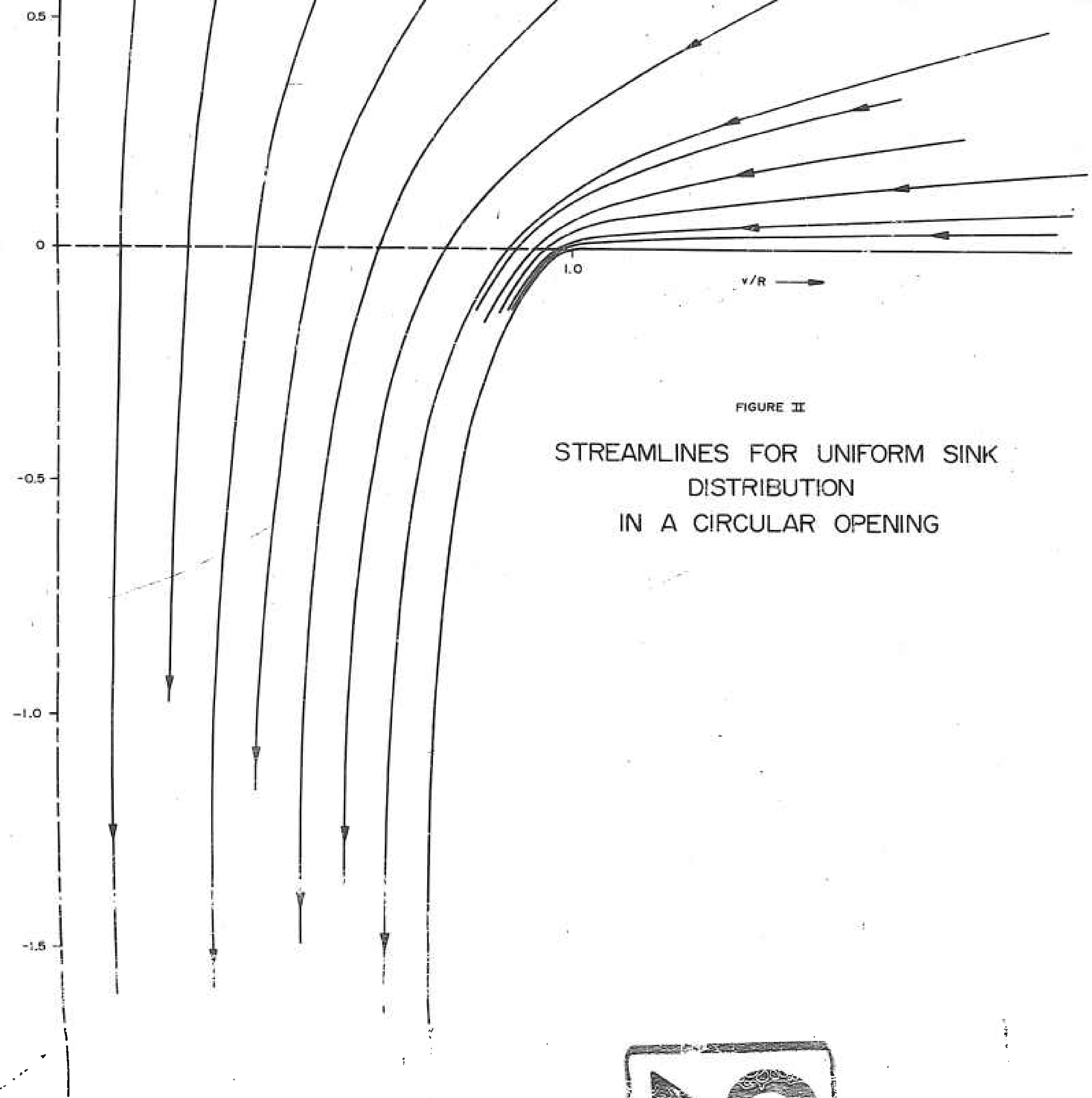


FIGURE II

STREAMLINES FOR UNIFORM SINK
DISTRIBUTION
IN A CIRCULAR OPENING